

On the Significance of Saturation Terms and Disease Induced Death on Seir Epidemic Model

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Abstract

The paper analyzed an SEIR epidemic model with disease induced death and saturation term. The existing model comprises of only Saturation terms and Treatment but disease induced death was absent in the model. The disease-free and endemic equilibria were also investigated to be asymptotically stable for Basic Reproduction Number to be less than unity. We investigated Basic Reproduction Number which was found using next generation matrix. Local and Global Stabilities were considered using linearization and Lyapunov function respectively. Numerical Simulation was done using Runge-Kutta of order 4 method. Our results show that disease induced death has no effect in disease eradication, other measures could be adopted.

Keywords: Basic Reproduction Number; Disease Free Equilibrium; Disease Induced Death; Saturation Terms.

Introduction

In recent paper of Adebimpe (2015) studied the Stability Analysis of an SIR epidemic model with Non-Linear Incidence Rate and Treatment. Kolawole and Olayiwola (2016) studied the behavioural analysis of a Susceptible-Expose-Infected-Recovered Susceptible (SEIRS) epidemic model, it was concluded that saturation term plays vital role in an SEIRS epidemic model. Also, (Ngna *et al.*, 2002 and Kolawole *et al.*, 2016) investigated simulation of epidemic models and concluded that disease die out if $R_0 < 1$. (Mojeeb *et al.*, 2017; Nita *et al.*, 2013; Combremaskel *et al.*, 2015; and Jial, 2011) considered the DFE and showed that it is locally asymptotically stable.

In this paper, a SEIR epidemic model with saturation terms and disease induced death was considered,

where $\frac{KSI}{1 + \alpha S^p + \beta I^q}$ is the nonlinear incidence rate with transmission rate K, and $p = q > 1$, p and q

being positive constant, α and β are the parameters which measure the effects of Sociological, Psychological effect or other mechanisms.

The Basic Mathematical Model

In this paper, model Adebimpe (2015) was adopted and modified by removing treatment term τ and incorporating compartment E and disease induced death m

The Existing model of Adebimpe (2015)

$$\begin{aligned}\frac{dS}{dt} &= b - dS - \frac{KSI}{1 + \alpha S^p + \beta I^q} + \gamma R \\ \frac{dI}{dt} &= \frac{KSI}{1 + \alpha S^p + \beta I^q} - (d + \mu)I - \tau \\ \frac{dR}{dt} &= \mu I - (d + \gamma)R + \tau\end{aligned}\quad (1)$$

Proposed Model

$$\begin{aligned}\frac{dS}{dt} &= b - dS - \frac{KSI}{1 + \alpha S^p + \beta I^q} + \gamma R \\ \frac{dE}{dt} &= \frac{KSI}{1 + \alpha S^p + \beta I^q} - (d + \mu)I \\ \frac{dI}{dt} &= \mu E - (d + \alpha + m)I \\ \frac{dR}{dt} &= \mu I - (d + \gamma)R\end{aligned}\quad (2)$$

Table 1: Description of the Used Parameters

Parameter	Description
b	Recruitment rate
α, β	Saturation term
p, q	Positive constant greater than one
d	Natural death rate
m	Disease induce death rate
γ	Rate at which recovered individual lose immunity and return to the susceptible class
μ	Natural recovery rate of the infective individual

Disease-Free Equilibrium (DFE)

The proposed model in equation (2) is therefore solved for the following.

At disease-free equilibrium $I = 0, E = 0, R = 0$, hence

$$b - dS - \frac{\beta SI}{1 + \alpha S^p + \beta I^q} + \gamma R = 0$$

$$b - dS = 0$$

$$S = \frac{b}{d}$$

Therefore,

$$(S, E, I, R) = (\frac{b}{d}, 0, 0, 0) \tag{3}$$

Endemic Equilibrium

At endemic equilibrium, $I \neq 0, p = q = 2$, hence SEIR=(S*E*I*R*)

To get R^* , Let

$$\mu I - (d + \gamma)R = 0$$

$$\mu I = (d + \gamma)R$$

$$R^* = \frac{\mu}{d + \gamma} \left(\frac{KS^* \pm \sqrt{(KS^*)^2 - 4\beta(b - dS^* + \gamma R^*)^2(1 + \alpha S^{*2})}}{2\beta(b - dS^* + \gamma R^*)} \right) \tag{3.1}$$

To get S^* , let

$$\frac{KSI}{1 + \alpha S^p + \beta I^q} - (d + \mu)I = 0$$

$$KS = (1 + \alpha S^p + \beta I^q)(d + \mu)$$

$$KS = (1 + \alpha S^2 + \beta I^2)(d + \mu)$$

$$KS - (\alpha S^2)(d + \mu) = (1 + \beta I^2)(d + \mu)$$

$$S^* = \frac{K \pm \sqrt{(K)^2 - 4\alpha(d + \mu)(1 + \beta I^{*2})}}{2\alpha(d + \mu)} \tag{3.2}$$

To get I^* , let

$$\begin{aligned}
 b - dS - \frac{KSI}{1 + \alpha S^p + \beta I^q} + \gamma R &= 0 \\
 b - dS + \gamma R &= \frac{KSI}{1 + \alpha S^2 + \beta I^2} \\
 (b - dS + \gamma R)(1 + \alpha S^2 + \beta I^2) &= KSI \\
 \beta I^2(b - dS + \gamma R) + (b - dS + \gamma R)(1 + \alpha S^2) &= KSI \\
 (b - dS + \gamma R)(1 + \alpha S^2) &= KSI - \beta I^2(b - dS + \gamma R) \\
 I^* &= \frac{KS^* \pm \sqrt{(KS^*)^2 - 4\beta(b - dS^* + \gamma R^*)^2(1 + \alpha S^{*2})}}{2\beta(b - dS^* + \gamma R^*)} \tag{3.3}
 \end{aligned}$$

To get E^* , let

$$\begin{aligned}
 \mu E - (d + \alpha + m)I &= 0 \\
 \mu E &= (d + \alpha + m)I \\
 E^* &= \frac{(d + \alpha + m)}{\mu} \left(\frac{KS^* \pm \sqrt{(KS^*)^2 - 4\beta(b - dS^* + \gamma R^*)^2(1 + \alpha S^{*2})}}{2\beta(b - dS^* + \gamma R^*)} \right) \tag{3.4}
 \end{aligned}$$

Basic Reproduction Number R_0

Using next generation matrix,

i.e. $R_0 = F.V^{-1}$

$$F = \begin{bmatrix} \frac{Kbd^p}{d(d^p + \alpha b^p)} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & (d + \mu) \\ (d + \alpha + m) & -\mu \end{bmatrix}$$

The inverse of V is obtained as

$$V^{-1} = \frac{-1}{(d + \mu)(d + \alpha + m)} \begin{bmatrix} -\mu & (d + \mu) \\ (d + \alpha + m) & 0 \end{bmatrix}$$

Hence,

$$\begin{aligned}
 R_0 = F.V^{-1} &= \begin{bmatrix} \frac{Kbd^p}{d(d^p + \alpha b^p)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\mu}{(d + \mu)(d + \alpha + m)} & \frac{-1}{(d + \alpha + m)} \\ \frac{-1}{(d + \mu)} & 0 \end{bmatrix} \\
 R_0 &= \begin{bmatrix} \frac{Kbd^p \mu}{d(d^p + \alpha b^p)(d + \mu)(d + \alpha + m)} & \frac{-Kbd^p}{d(d^p + \alpha b^p)(d + \alpha + m)} \\ 0 & 0 \end{bmatrix}
 \end{aligned}$$

The equation with the dominant Eigen value becomes the basic reproduction number

$$R_0 = \frac{Kbd^p \mu}{d(d^p + \alpha b^p)(d + \mu)(d + \alpha + m)} \tag{4}$$

Local Stability of Disease-Free Equilibrium

$$\frac{dS}{dt} = b - dS - \frac{KSI}{1 + \alpha S^p + \beta I^q} + \gamma R$$

$$\frac{dE}{dt} = \frac{KSI}{1 + \alpha S^p + \beta I^q} - (d + \mu)E$$

$$\frac{dI}{dt} = \mu E - (d + \alpha + m)I$$

$$\frac{dR}{dt} = \alpha I - (d + \gamma)R$$

The Jacobian matrix becomes,

$$J(E_0) = \begin{bmatrix} -d & 0 & -(d + \mu)(d + \alpha + m)R_0 & \gamma \\ 0 & -(d + \mu) & (d + \mu)(d + \alpha + m)R_0 & 0 \\ 0 & \mu & -(d + \alpha + m) & 0 \\ 0 & 0 & \alpha & -(d + \gamma) \end{bmatrix} \tag{5}$$

then

$$J(E_0) = \begin{vmatrix} -d - \lambda & 0 & -(d + \mu)(d + \alpha + m)R_0 & \gamma \\ 0 & -(d + \mu) - \lambda & (d + \mu)(d + \alpha + m)R_0 & 0 \\ 0 & \mu & -(d + \alpha + m) - \lambda & 0 \\ 0 & 0 & \alpha & -(d + \gamma) - \lambda \end{vmatrix} = 0 \tag{6}$$

$$\left[\lambda^4 + [(4d + \alpha + m + \gamma + \mu)]\lambda^3 + [(d + \mu)(2d + \alpha + m + \gamma)(d + \mu + \gamma)(d + \alpha + m) - (d + \mu)(d + \gamma) - \mu]\lambda^2 + d[(d + \mu)(2d + \alpha + m + \gamma) + (d + \alpha + m)(d + \gamma) + \mu + (d + \mu)(d + \alpha + m)(d + \gamma + R_0)]\lambda + [d(3d + \alpha + m + \gamma + \mu) + d(d + \mu)(d + \alpha + m)(d + \gamma + R_0) + \mu(d + \gamma)] \right] = 0$$

Let

$$A = [(4d + \alpha + m + \gamma + \mu)]$$

$$B = [(d + \mu)(2d + \alpha + m + \gamma)(d + \mu + \gamma)(d + \alpha + m) - (d + \mu)(d + \gamma) - \mu]$$

$$C = d[(d + \mu)(2d + \alpha + m + \gamma) + (d + \alpha + m)(d + \gamma) + \mu + (d + \mu)(d + \alpha + m)(d + \gamma + R_0)]$$

$$D = [d(3d + \alpha + m + \gamma + \mu) + d(d + \mu)(d + \alpha + m)(d + \gamma + R_0) + \mu(d + \gamma)]$$

Then,

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0$$

Using the Routh-Hurwitz criterion, it can be seen that all the Eigen values of the characteristics equation above have negative real part if and only if:

$$A > 0, B > 0, C > 0, D > 0, ABC - D > 0 \tag{7}$$

Therefore, it is locally asymptotically stable if and only if inequalities in [7] are satisfied

Global Stability of Disease-Free Equilibrium

We consider the Lyapunov function,

$$V = \mu E + (d + \mu)I$$

$$\frac{dV}{dt} = \mu \frac{dE}{dt} + (d + \mu) \frac{dI}{dt}$$

$$\frac{dV}{dt} = \mu \left[\frac{KSI}{1 + \alpha S^p + \beta I^q} - (d + \mu)E \right] + (d + \mu)[\mu E - (d + \alpha + m)I]$$

$$\frac{dV}{dt} = (d + \mu)(d + \alpha + m) \left[\frac{\mu K b d^p}{d(d^p + \alpha b^p)(d + \mu)(d + \alpha + m)} - 1 \right] I$$

$$(d + \mu)(d + \alpha + m)[R_0 - 1] \leq 0 \tag{8}$$

Hence the disease-free equilibrium is globally asymptotically stable as seen in equation (8).

Local Stability of Endemic Equilibrium

If we let

$$\begin{aligned}
 S - S^* &= w, E - E^* = x, I - I^* = y, R - R^* = z \\
 \frac{ds}{dt} &= \frac{dw}{dt}, \frac{dE}{dt} = \frac{dx}{dt}, \frac{dI}{dt} = \frac{dy}{dt}, \frac{dR}{dt} = \frac{dz}{dt} \\
 \frac{dw}{dt} &= b - d(w + S^*) - \frac{K(w + S^*)(y + I^*)}{1 + \alpha S^p + \beta I^q} + \gamma(z + R^*) \\
 \frac{dx}{dt} &= \frac{K(w + S^*)(y + I^*)}{1 + \alpha S^p + \beta I^q} - (d + \mu)(x + E^*) \\
 \frac{dy}{dt} &= \mu(x + E^*) - (d + \mu + m)(y + I^*) \\
 \frac{dz}{dt} &= \alpha(y + I^*) - (d + \gamma)(z + R^*) \tag{9} \\
 \frac{dw}{dt} &= -dw - \frac{KwI^*}{1 + \alpha S^p + \beta I^q} - \frac{KyS^*}{1 + \alpha S^p + \beta I^q} + \gamma z + \text{non linear terms} \\
 \frac{dx}{dt} &= \frac{KwI^*}{1 + \alpha S^p + \beta I^q} + \frac{KyS^*}{1 + \alpha S^p + \beta I^q} - (d + \mu)x + \text{non linear terms} \\
 \frac{dy}{dt} &= \mu x - (d + \mu + m)y + \text{non linear terms} \\
 \frac{dz}{dt} &= \alpha y - (d + \gamma)z + \text{non linear terms} \tag{10}
 \end{aligned}$$

The Jacobian matrix becomes

$$J(E^*) = \begin{bmatrix} \left(-d - \frac{KI^*}{1 + \alpha S^p + \beta I^q} \right) & 0 & -\frac{KS^*}{1 + \alpha S^p + \beta I^q} & \gamma \\ \frac{KI^*}{1 + \alpha S^p + \beta I^q} & -(d + \mu) & \frac{KS^*}{1 + \alpha S^p + \beta I^q} & 0 \\ 0 & \mu & -(d + \alpha + m) & 0 \\ 0 & 0 & \mu & -(d + \gamma) \end{bmatrix}$$

Equation (10) beomes

$$(-d - A_1 - \lambda)[(-d + \mu) - \lambda](d + \alpha + m - \lambda)((d + \gamma) - \lambda) - \mu(d + \gamma) - \lambda + [-A_2 A_1(-\mu(d + \gamma) - \lambda)] - \gamma A_1(\mu^2 \alpha) = 0$$

By expansion, we have

$$\begin{aligned}
 &\lambda^4 + [(4d + \alpha + m + \gamma + 2\mu + A_1)]\lambda^3 + [(d + \mu)(d + A_1) + (2d + \alpha + m + \gamma + \mu)(2d + \mu + A_1) + \\
 &(d + \gamma)(d + \alpha + m + \mu)]\lambda^2 + [(d + A_1)(d + \mu)(2d + \alpha + m + \gamma + \mu) + (d + A_1)(d + \gamma)(d + \alpha + m + \mu) + \\
 &(d + \gamma)(d + \mu)(d + \alpha + m + \mu) + A_1 A_2 \mu]\lambda \\
 &+ [(d + \gamma)(d + \mu)(d + \alpha + m + \mu)(d + A_1) + A_1 A_2 \mu(d + \gamma) - A_1 \gamma \mu^2] = 0
 \end{aligned}$$

Let

$$\begin{aligned}
 a_0 &= (4d + \alpha + m + \gamma + 2\mu + A_1) \\
 a_1 &= (d + \mu)(d + A_1) + (2d + \alpha + m + \gamma + \mu)(2d + \mu + A_1) + (d + \gamma)(d + \alpha + m + \mu) \\
 a_2 &= (d + A_1)(d + \mu)(2d + \alpha + m + \gamma + \mu) + (d + A_1)(d + \gamma)(d + \alpha + m + \mu) + (d + \gamma) \\
 &\quad (d + \mu)(d + \alpha + m + \mu) + A_1 A_2 \mu \\
 a_3 &= (d + \gamma)(d + \mu)(d + \alpha + m + \mu)(d + A_1) + A_1 A_2 \mu(d + \gamma) - A_1 \gamma \mu^2
 \end{aligned}$$

Therefore,

$$\lambda^4 + a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0$$

Using the Routh-Hurwitz criterion, it can be seen that all the Eigen values of the characteristics equation above have negative real part if and only if:

$$a_0 > 0, a_1 > 0, a_2 > 0, a_3 > 0, a_0 a_1 a_2 - a_3 > 0 \tag{12}$$

Hence, it is locally asymptotically stable if and only if inequalities in (12) are satisfied.

Results and Discussion

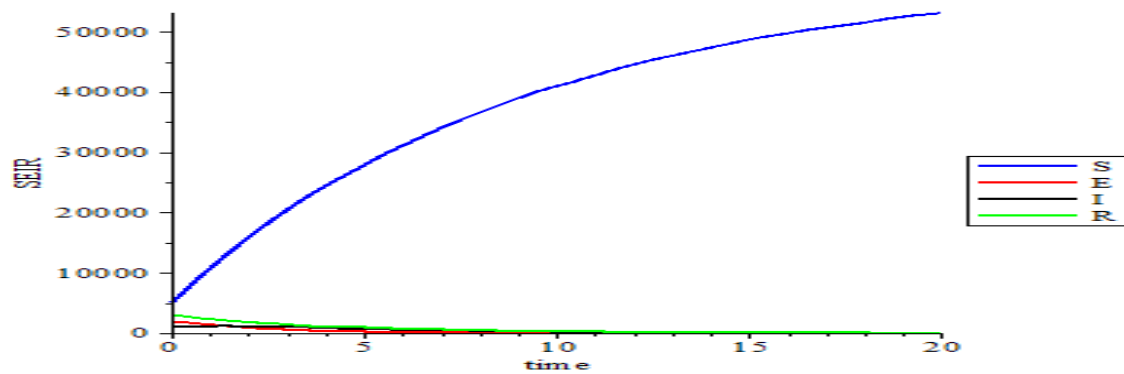


Figure 1: Graph of Susceptible (S), Exposed (E), Infected (I), Recovered (R) against time (t) with $\beta = 0.2$, $\Lambda = 6000$, $\mu = 0.3$, $d = 0.1$, $m = 0.1$, $\alpha = 0.1$, $p = 2$, $q = 2$, $\gamma = 0.2$

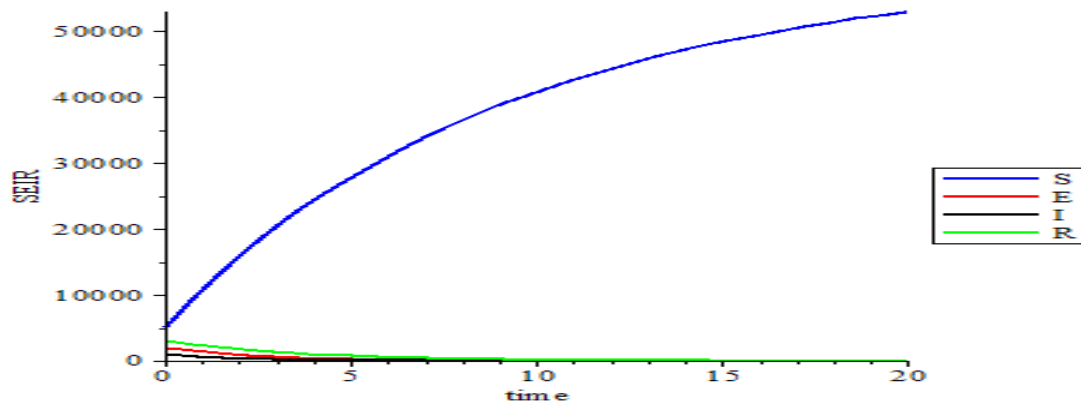


Figure 2: Graph of Susceptible (S), Exposed (E), Infected (I), Recovered (R) against time (t) with $\beta = 0.2$, $\Lambda = 6000$, $\mu = 0.3$, $d = 0.1$, $m = 1$, $\alpha = 0.1$, $p = 2$, $q = 2$, $\gamma = 0.2$

Figure 1 reveals the effect of saturation terms on the model when $m=0.1$. It was observed that at lower induced death and saturation term, the susceptible class increase drastically. Also, Figure 2 reveals the effect of saturation term on the model when $m=1.0$. it was observed that the susceptible class increases to the highest level and infected class reduces to the lowest level.

Conclusions

In this paper, it was shown that disease induced death has no significant role to play in disease eradication. The model of Adebimpe (2015) has both saturation terms including treatment. The simulation shows that disease induced death is not a better measure for disease eradication. Other measures like treatment and vaccination including saturation term are better.

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