

On the Daftardar and Jafari Method for the Numerical Solution of Duffing Equations

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Abstract

In this study, the Daftardar Gejji and Hossein Jafari Method (DJM) is used to solve Duffing differential equations. Some numerical examples are presented to demonstrate the applicability of the method. The results obtained show that the method is computationally reliable and effective. Maple 18 software was used for the implementation.

Keywords: DJM; Duffing Differential Equation; Damped Equation.

Introduction

Over many recent decades, many physical phenomena have been modeled using nonlinear ordinary differential equations. One of these equations is called Duffing equation (or Duffing oscillator), which has received remarkable attention due to its classical applications in all areas of sciences and engineering. It is named after electrical engineer German Georg Duffing in Sheu *et al.* (2007). Duffing equation occurs as result of the motion of a body subjected to a nonlinear spring power, linear sticky damping, and periodic powering. Oscillations of mechanical systems under the action of a periodic external force can be revealed by using Duffing equation, Agadjanov (2006). Recently, the quasi-two-step method for the nonlinear undamped duffing equation is presented by Wang (2005). A direct method to find the exact solution to the damped duffing equation and travelling wave solutions to the reaction- diffusion equation was used by Feng (2006). Duffing equation helps in model of periodic orbit extraction, nonlinear mechanical oscillators and prediction of diseases Balaji (2014). On the other hand, Duffing differential equations have been effectively dealt in many works, Pezeshki and Dowell (1987).

The general form of Duffing equation is considered in this paper. The equation is of the following form Agadjanov (2006).

$$u''(x) + k_1 u'(x) + k_2 u(x) + k_3 u^3(x) = f(x) \quad (1)$$

With initial conditions,

$$u(0) = a, \text{ and } u'(0) = a$$

Where k_1, k_2, k_3 a and b are real constants and $f(x)$ is real-valued function.

The k_1 controls the size of damping, k_2 controls of size stiffness, k_3 controls of size amount of non-linearity in the restoring force. Numerous works has been done on the development of efficient methods for simulating Duffing equation such as variational iteration method, He *et al.* (2010), homotopy perturbation transform method, Khan and Wu (2011), improved Taylor matrix method, Berna and Mehmet (2013), differential transform method, Tabatabaei and Gunerhan (2014), restarted Adomain decomposition method as discussed in Vahidi *et al.* (2012) among others. In 2006, a new iterative method was proposed by Daftardar and Jafari (2006, 2011). This method had been applied to many equations such as ordinary differential equation, algebraic equations, partial differential equations and integral equations. In this paper, the main aim is to implement DJM for solving Duffing equations and to also develop an iterative computational method for simulating the Duffing equations.

Daftardar and Jafari Method (DJM)

Consider the following general functional equation, Daftardar and Jafari (2006).

$$u = N(u) + f, \quad (2)$$

Where N is nonlinear operator from a Banach space $B \rightarrow B$, f is a known function, we want to find the solution of (2) having the series of the form:

$$u = \sum_{i=0}^{\infty} u_i, \tag{3}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=0}^{\infty} \left\{ N\left(\sum_{j=0}^i u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\} \tag{4}$$

From Eqs. (3) and (4), Eq. (1) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=0}^{\infty} \left\{ N\left(\sum_{j=0}^i u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\} \tag{5}$$

Therefore, we define the recurrence relation:

$$\begin{cases} u_0 = f, \\ u_1 = N(u_0) \\ u_{m+1} = N(u_0 + \dots + u_m) - N(u_0 + \dots + u_{m-1}), m = 1, 2, \dots \end{cases} \tag{6}$$

Then

$$u = f + \sum_{i=0}^{\infty} u_i, \tag{7}$$

The series absolutely and uniformly converges to a solution of Eq. (2), according to Daftardar and Jafari (2006).

Application of DJM for Solving Duffing Equation

According to [14], let us consider a form of Duffing equation given in Eq. (1).

Eq. (1) can be written in an operator form as:

$$L_{xx}u(x) + k_1L_xu(x) + k_2u(x) + k_3u^3(x) = f(x) \tag{8}$$

Where $L_{xx} = \frac{d^2}{dx^2}$, $L_x = \frac{d}{dx}$

Now, Let assume that the inverse operator L_x^{-1} and L_{xx}^{-1} exist and it can be taken with respect x from 0 to x

$$L_{xx}^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx dx, L_x^{-1}(\cdot) = \int_0^x (\cdot) dx \tag{9}$$

Then, taking the inverse operator L_{xx}^{-1} to both sides of the equation (8), we have:

$$L_{xx}^{-1}L_{xx}u(x) + k_1L_{xx}^{-1}L_xu(x) + L_{xx}^{-1}(k_2u(x) + k_3u^3(x)) = L_{xx}^{-1}f(x) \tag{10}$$

Solving equation equation (10) with initial condition, we obtain:

$$\begin{aligned} u(x) - a - bx + L_x^{-1}k_1u(x) - k_1ax + L_{xx}^{-1}(k_2u(x) + k_3u^3(x)) &= g(x) \\ u(x) = a + k_1ax + bx + g(x) - L_x^{-1}k_1u(x) - L_{xx}^{-1}(k_2u(x) + k_3u^3(x)) \end{aligned} \tag{11}$$

where, $g(x) = \int_0^x \int_0^x f(t) dt dt$

Now, we convert L_{xx}^{-1} from double integral to single integral from the relation, Al-Jawaryet al. (2016):

$$\int_0^x \int_0^x \cdots \int_0^{x_{n-1}} u(x_n) dx_n dx_{n-1} \cdots dx_1 = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} u(t) dt \tag{12}$$

Then for the double integral:

$$L_{x_x}^{-1} = \int_0^x \int_0^x u(t) dt dt = \int_0^x (x-t)u(t) dx \tag{13}$$

By using Eqs. (6) and (7), we obtain the following components:

$$u_0(x) = a + k_1 ax + bx + g(x) \tag{14}$$

$$u_1(x) = N(u_0) = -\int_0^x (k_1(u_0)(t))dt - \int_0^x (x-t)(k_2 u_0(t) + k_3 u_0^3(t))dt \tag{15}$$

$$u_2(x) = N(u_0 + u_1) - u_1,$$

$$u_2(x) = -\int_0^x (k_1(u_0 + u_1)(t))dt - \int_0^x (x-t)(k_2(u_0 + u_1)(t) + k_3(u_0 + u_1)^3(t))dt - u_1, \tag{16}$$

$$u_3(x) = -\int_0^x (k_1(u_0 + u_1 + u_2)(t))dt - \int_0^x (x-t)(k_2(u_0 + u_1 + u_2)(t) + k_3(u_0 + u_1 + u_2)^3(t))dt + \int_0^x (k_1(u_0 + u_1)(t))dt + \int_0^x (x-t)(k_2(u_0 + u_1)(t) + k_3(u_0 + u_1)^3(t))dt \tag{17}$$

$$u_4(x) = -\int_0^x (k_1(u_0 + u_1 + u_2 + u_3)(t))dt - \int_0^x (x-t)(k_2(u_0 + u_1 + u_2 + u_3)(t) + k_3(u_0 + u_1 + u_2 + u_3)^3(t))dt + \int_0^x (k_1(u_0 + u_1 + u_2)(t))dt + \int_0^x (x-t)(k_2(u_0 + u_1 + u_2)(t) + k_3(u_0 + u_1 + u_2)^3(t))dt \tag{18}$$

By continuing in this technique, the (n+1)th approximation of the exact solution for the unknown function u(x) can be achieved as:

$$u_{n+1}(x) = N(u_0 + u_1 + \dots + u_n) - N(u_0 + u_1 + \dots + u_{n-1}) \tag{19}$$

$$u_{n+1}(x) = -\int_0^x k_1(u_0 + \dots + u_n)(t)dt - \int_0^x (x-t)(k_2(u_0 + \dots + u_n)(t) + k_3(u_0 + \dots + u_n)^3(t))dt + \int_0^x (k_1(u_0 + \dots + u_{n-1})(t))dt + \int_0^x (x-t)(k_2(u_0 + \dots + u_{n-1})(t) + k_3(u_0 + \dots + u_{n-1})^3(t))dt \tag{20}$$

Then,

$$u(x) = \sum_{n=0}^{\infty} u_n. \tag{21}$$

Illustrative Examples

Problem 1: We consider the damping Duffing equation of the form:

$$u''(x) + u'(x) + u(x) + u^3(x) = 4 + 6x + 6x^2 + 5x^3 \tag{22}$$

With initial conditions $u(0) = 1, u'(0) = 1$

The exact solution is given by:

$$u(x) = e^x \tag{23}$$

Applying the initial condition in the Eq. (22), we have

$$u(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^3 + 2x^2 + 2x + 1 - \int_0^x u(t)dt - \int_0^x (x-t)(u(t) + u^3(t))dt \tag{24}$$

Using DJM algorithm, we obtain the following series:

$$u_0(x) = 1 + 2x + 2x^2 + x^3 + \frac{1}{2}x^4 + \frac{1}{4}x^5 \tag{25}$$

$$u_{n+1}(x) = -\int_0^x (u_0 + \dots + u_n)(t)dt - \int_0^x (x-t)((u_0 + \dots + u_n)(t) + (u_0 + \dots + u_n)^3(t))dt + \int_0^x (u_0 + \dots + u_{n-1})(t)dt + \int_0^x (x-t)((u_0 + \dots + u_{n-1})(t) + (u_0 + \dots + u_{n-1})^3(t))dt \tag{26}$$

$$u_1(x) = -\int_0^x (u_0)(t)dt - \int_0^x (x-t)((u_0)(t) + (u_0^3)(t))dt \tag{27}$$

$$u_1(x) = -x - 2x^2 - 2x^3 - \frac{23}{24}x^4 - \frac{19}{10}x^5 - \frac{41}{24}x^6 - \frac{55}{42}x^7 - \dots$$

$$u_2(x) = -\int_0^x (u_0 + u_1)(t)dt - \int_0^x (x-t)((u_0 + u_1)(t) + (u_0 + u_1)^3(t))dt - u_1$$

$$u_2(x) = \frac{1}{2}x^2 + \frac{4}{3}x^3 + \frac{23}{12}x^4 + \frac{7}{3}x^5 + \frac{427}{180}x^6 + \frac{1667}{840}x^7 + \dots \tag{28}$$

$$u_3(x) = -\int_0^x (u_0 + u_1 + u_2)(t)dt - \int_0^x (x-t)((u_0 + u_1 + u_2)(t) + (u_0 + u_1 + u_2)^3(t))dt + \int_0^x (u_0 + u_1)(t)dt + \int_0^x (x-t)((u_0 + u_1)(t) + (u_0 + u_1)^3(t))dt \tag{29}$$

$$u_3(x) = -\frac{1}{6}x^3 - \frac{1}{4}x^4 - \frac{4}{5}x^5 + \frac{71}{72}x^6 - \frac{2449}{2520}x^7 - \dots$$

and so on.

By continuing in this technique, the unknown functions $u(x)$ can be achieved as:

$$u(x) = u_0 + u_1 + u_2 + u_3 + \dots,$$

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots \tag{30}$$

This has the closed form:

$$u(x) = e^x$$

Which is the exact solution of Eq.(22)

Problem 2: We consider the damping Duffing equation:

$$u''(x) + 5u'(x) + u(x) + 3u^3(x) = 3x^6 + x^2 + 10x + 2 \tag{31}$$

With initial conditions $u(0) = 0, u'(0) = 0$

The exact solution is given by:

$$u(x) = x^2 \tag{32}$$

Applying the DJM with initial conditions, we obtain the following:

$$u(x) = \frac{3}{56}x^8 + \frac{1}{12}x^4 + \frac{5}{3}x^3 + x^2 - 5\int_0^x u(t)dt - \int_0^x (x-t)(u(t) + 3u^3(t))dt \tag{33}$$

We obtain the following components by using DJM

$$u_0(x) = x^2 + \frac{5}{3}x^3 + \frac{1}{12}x^4 + \frac{3}{56}x^8 \tag{34}$$

$$u_{n+1}(x) = -5 \int_0^x (u_0 + \dots + u_n)(t) dt - \int_0^x (x-t)((u_0 + \dots + u_n)(t) + 3(u_0 + \dots + u_n)^3(t)) dt$$

$$+ 5 \int_0^x (u_0 + \dots + u_{n-1})(t) dt + \int_0^x (x-t)((u_0 + \dots + u_{n-1})(t) + 3(u_0 + \dots + u_{n-1})^3(t)) dt$$

$$u_1(x) = -5 \int_0^x (u_0)(t) dt - \int_0^x (x-t)((u_0)(t) + 3(u_0^3)(t)) dt$$

$$u_1(x) = -\frac{5}{3}x^3 - \frac{13}{6}x^4 - \frac{1}{6}x^5 - \frac{1}{360}x^6 - \frac{3}{56}x^8 - \dots$$

$$u_2(x) = -5 \int_0^x (u_0 + u_1)(t) dt - \int_0^x (x-t)((u_0 + u_1)(t) + 3(u_0 + u_1)^3(t)) dt - u_1$$

$$u_2(x) = \frac{25}{12}x^4 + \frac{9}{4}x^5 + \frac{19}{90}x^6 + \frac{1}{168}x^7 + \frac{1}{20160}x^8 + \dots$$

$$u_3(x) = -5 \int_0^x (u_0 + u_1 + u_2)(t) dt - \int_0^x (x-t)((u_0 + u_1 + u_2)(t) + 3(u_0 + u_1 + u_2)^3(t)) dt +$$

$$+ 5 \int_0^x (u_0 + u_1)(t) dt + \int_0^x (x-t)((u_0 + u_1)(t) + 3(u_0 + u_1)^3(t)) dt,$$

$$u_3(x) = -\frac{25}{12}x^5 - \frac{35}{18}x^6 - \frac{103}{504}x^7 - \frac{151}{20160}x^8 - \dots$$

And so on.

We have the solution in a series form as:

$$u(x) = u_0 + u_1 + u_2 + u_3 + u_4 + \dots,$$

$$u(x) = x^2$$

Which is the exact solution of Eq (31).

Problem 3: Consider the damping Duffing equation of the form:

$$u''(x) + u'(x) + u(x) - u^3(x) = 3 + 6x + \frac{9}{2}x^2 + x^3$$

With initial conditions $u(0) = 0, u'(0) = 1$

The exact solution is given by:

$$u(x) = xe^x$$

Applying the DJM with initial conditions, we obtain the following:

$$u(x) = \frac{1}{20}x^5 + \frac{3}{8}x^4 + x^3 + \frac{3}{2}x^2 + x - \int_0^x u(t) dt - \int_0^x (x-t)(u(t) - u^3(t)) dt$$

Now, we apply the algorithm of the DJM, we have:

$$u_0(x) = x + \frac{3}{2}x^2 + x^3 + \frac{3}{8}x^4 + \frac{1}{20}x^5$$

$$u_{n+1}(x) = - \int_0^x (u_0 + \dots + u_n)(t) dt - \int_0^x (x-t)((u_0 + \dots + u_n)(t) - (u_0 + \dots + u_n)^3(t)) dt$$

$$+ \int_0^x (u_0 + \dots + u_{n-1})(t) dt + \int_0^x (x-t)((u_0 + \dots + u_{n-1})(t) - (u_0 + \dots + u_{n-1})^3(t)) dt$$

$$u_1(x) = -\int_0^x (u_0)(t)dt - \int_0^x (x-t)((u_0)(t) - (u_0^3)(t))dt$$

$$u_1(x) = -\frac{1}{2}x^2 - \frac{2}{3}x^3 - \frac{3}{8}x^4 - \frac{3}{40}x^5 + \frac{31}{240}x^6 + \frac{97}{420}x^7 + \dots$$
(45)

$$u_2(x) = -\int_0^x (u_0 + u_1)(t)dt - \int_0^x (x-t)((u_0 + u_1)(t) - (u_0 + u_1)^3(t))dt - u_1$$

$$u_2(x) = \frac{1}{6}x^3 + \frac{5}{24}x^4 + \frac{13}{120}x^5 - \frac{1}{40}x^6 - \frac{43}{280}x^7 - \dots$$
(46)

$$u_3(x) = -\int_0^x (u_0 + u_1 + u_2)(t)dt - \int_0^x (x-t)((u_0 + u_1 + u_2)(t) - (u_0 + u_1 + u_2)^3(t))dt +$$

$$+ \int_0^x (u_0 + u_1)(t)dt + \int_0^x (x-t)((u_0 + u_1)(t) - (u_0 + u_1)^3(t))dt$$
(47)

$$u_3(x) = -\frac{1}{24}x^4 - \frac{1}{20}x^5 - \frac{1}{40}x^6 + \frac{13}{1008}x^7 + \dots$$

And so on. The solution in a series form is given by:

$$u(x) = u_0 + u_1 + u_2 + u_3 + \dots,$$

$$u(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \dots$$
(48)

$$u(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$
(49)

$$u(x) = xe^x$$

Eq. (49) is the exact solution of Eq. (40).

Discussion of Results

In the section, we compare our approximate solution obtained using DJM and exact solution. In comparison, we see that DJM is converges faster.

Table 1: Comparison of DJM and Exact solution for different values of x for Problem 1 when n=2

x	u(Exact)	u(DJM)	Error (DJM)
0	1	1	0
0.1	1.105171	1.105171	1.41E-09
0.2	1.221403	1.221403	9.15E-08
0.3	1.349859	1.349858	1.06E-06
0.4	1.491825	1.491819	6.03E-06
0.5	1.648721	1.648698	2.34E-05
0.6	1.822119	1.822048	7.08E-05
0.7	2.013753	2.013571	1.81E-04
0.8	2.225541	2.225131	4.10E-04
0.9	2.459603	2.458758	8.45E-04
1.0	2.718282	2.716667	1.62E-03

Table 2: Comparison of DJM and Exact solution for different values of x for Problem 3 when n=2

x	u(Exact)	u(DJM)	Error (DJM)
0	0	0	0
0.1	0.110517092	0.110517083	8.474230E-09
0.2	0.244280552	0.244280000	5.516320E-07
0.3	0.404957642	0.404951250	6.392270E-06
0.4	0.596729879	0.596693333	3.654570E-05
0.5	0.824360635	0.824218750	1.418850E-04
0.6	1.093271280	1.092840000	4.312800E-04
0.7	1.409626895	1.408519583	1.107312E-03
0.8	1.780432743	1.777920000	2.512743E-03
0.9	2.213642800	2.208453750	5.189050E-03
1.0	2.718281828	2.708333333	9.948495E-03

Conclusions

In this paper, we were able to solve Duffing equations by applying DJM. The numerical solutions were obtained without any restrictive assumptions for nonlinear terms as required by some existing techniques. Also, by solving some examples, the rate at which DJM converges is remarkable. The DJM appears to be very accurate to employ with reliable results.

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