

Some New Features of a Complete Generalized Metric Space

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Abstract

In this paper, existence and uniqueness of fixed points of operators satisfying certain contractive conditions in a complete G -metric space are stated and proved. Furthermore, the stability of Picard iterative procedure is investigated for mappings T satisfying some fairly general contractive conditions in a complete G -metric space.

Keywords: Complete; Mapping; Stability; Iteration.

Introduction

Rhoades (1977) gave comparison of various definitions of contractive mappings and defined contractive type mappings on a complete metric space X which are generalizations of the well-known Banach contraction, and have the property that such mappings have unique fixed point in X . Osileke (1996) established some stability results for fixed point iteration procedures in a complete metric space. Mustafa and Sims (2006) introduced a new notion of generalized metric space called G -metric space, after proving that most of the results concerning the topological properties of D -metric space were incorrect. To repair this setback, a more appropriate notion of a generalized metrics, called G -metric space was defined and new properties for G -metric space were proved. Furthermore, the relationship between metric space and G -metric space was established. Zead Mustafa *et al.* (2010) proved some fixed point results for mapping satisfying sufficient conditions in a complete G -metric space, also that if the G -metric space (X, G) is symmetric, then the existence and uniqueness of these fixed point results follow from well-known theorems in usual metric space (X, d_G) , where (X, d_G) is the usual metric space defined from G -metric space (X, G) . Mujahid Abbas (2012) worked on fixed and related fixed point theorems for three maps in G -metric space, unique common fixed points of three maps that satisfy a generalized (ϕ, ψ) -weak contractive condition are obtained. It is noted that the existence of a fixed point of any one of the mappings implies that the three mappings have a common fixed point. Agarwal *et al.* (2013) Gave some remarks on coupled fixed point theorems in G -metric spaces. Agarwal *et al.* (2015) gave new definitions, properties, lemmas and theorems regarding metric type spaces. Rauf *et al.* (2017) studied some fixed point theorems for contractive conditions in a G -metric space.

In this paper fixed point results for operators satisfying some more general contractive conditions are considered and also stability theorems for some additional iteration procedures in a G -metric space were proved.

Definitions of Some Terms

Definition 2.1 (Mustafa and Sims, 2006):

Let X be a nonempty set, and let $G: X \times X \times X \rightarrow R^+$, be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, (symmetry in all three variables); and

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality)

then the function G is called a generalized metric, or, more specifically a G -metric on X , and the pair (X, G) is called a G -metric space.

Proposition 2.1(Mustafa and Sims, 2006):

Let (X, G) be G -metric space. Then the following are equivalent.

1. (x_n) is G-convergent to x . for a function $f : X \rightarrow X$
2. $G(x_n, x_n, x) = 0, as n \rightarrow \infty$.
3. $G(x_n, x, x) = 0, as n \rightarrow \infty$.
4. $G(x_m, x_n, x) = 0, as m, n \rightarrow \infty$.

Lemma 2.1(Agarwal et al., 2015):

Let (X, G) be a G -metric space. Then, for any $x, y, z, a \in X$ the following properties hold.

1. $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$.
2. $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.
3. $|G(x, y, z) - G(x, y, a)| \leq \max\{G(a, z, z), G(z, a, a)\}$.
4. If $n \geq 2$ and $x_1, x_2, \dots, x_n \in X$ then

$$G(x_1, x_1, x_n) \leq \sum_{i=1}^{n-1} G(x_i, x_i, x_{i+1})$$
5. If $G(x, y, z) = 0$, then $x = y = z$
6. $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.
7. $G(x, y, z) \leq \frac{2}{3}[G(x, y, a) + G(x, a, z) + G(a, y, z)]$;
8. $|G(x, y, z) - G(x, y, a)| \leq G(a, x, z)$; and
9. $G(x, y, y) \leq 2G(x, y, z)$.

Lemma 2.2(Agarwal et al., 2015):

Let $\{x_n\}$ be a sequence in G-metric space (X, G) and assume that there exist a constant $\lambda \in [0, 1)$ and $n_0 \in N$ such that, at least, one of the following conditions holds:

1. $G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda(G(x_n, x_{n+1}, x_{n+1}))$ for all $n \geq n_0$
2. $G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda(G(x_n, x_n, x_{n+1}))$ for all $n \geq n_0$

Then $\{x_n\}$ is a Cauchy sequence in (X, G) .

Lemma 2.3(Mustafa and Sims, 2006):

If (X, G) is a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables, that is, if $x, y, z \in X$ and $\{x_n\}, \{y_n\}, \{z_n\} \subseteq X$ are sequence in X such that

$\{x_m\} \rightarrow x, \{y_m\} \rightarrow y$ and $\{z_m\} \rightarrow z$ then $\{G(x_m, y_m, z_m)\} \rightarrow G(x, y, z)$.

Proposition 2.2(Agarwal et al., 2015):

Let $\{x_n\}$ be a sequence in G-metric space (X, G) .

1. If the following conditions holds:
for all $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$G(x_n, x_m, x_m) \leq \varepsilon, \text{ for all } m, n \geq n_0$$
2. If the following condition holds:
for all $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$G(x_n, x_n, x_m) \leq \varepsilon, \text{ for all } m, n \geq n_0$$

then $\{x_n\}$ is a Cauchy sequence in (X, G) .

Theorem 2.1(Rauf et al., 2017)

Let (X, G) be a complete G -metric space, and let $T : X \rightarrow X$ be a mapping satisfying the following condition

$$G(Tx, Ty, Tz) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(x, y, z) \quad (1)$$

for all $x, y, z \in X$ where $0 \leq a + b + c \leq 1$. Then T has a unique fixed point (say x^*)

$$Tx^* = x^*.$$

Theorem 2.2(Rauf et al. 2017)

Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying the following condition

$$G(Tx, Ty, Tz) \leq \alpha G(x, Tx, Tx) + \psi G(x, y, z) \quad (2)$$

for all $x, y, z \in X$ where $0 \leq \alpha + \psi \leq 1$, then T has a unique fixed point x^* in X ; $Tx^* = x^*$.

Results

Theorem 3.1 Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying the following condition,

$$\begin{aligned} G(Tx, Ty, Tz) &\leq a_1 G(x, y, z) + a_2 G(x, Tx, Tx) + a_3 G(y, Ty, Ty) \\ &\quad + a_4 G(x, Ty, Ty) + a_5 G(y, Tx, Tx) \end{aligned} \quad (3)$$

for all $x, y, z \in X$ where each a_i are nonnegative constants satisfying

$$\sum_{i=1}^5 a_i < 1 \quad \text{and} \quad a_i \leq a_{i+1} \quad (4)$$

then T has a unique fixed point in X .

Proof

Let $x_0 \in X$ be an arbitrary point. Define the sequence $\{x_n\} \in X$ by

$$x_{n+1} = Tx_n \quad \text{for all } n \in N \quad (5)$$

T is a Picard sequence of $\{x_n\}$ based on x_0 . If there is $n_0 \in N$ such that $x_{n_0+1} = x_{n_0}$, then

x_{n_0} is a fixed point of T .

Assume $x_{n+1} \neq x_n$

Let $x = x_n, y = z = x_{n+1}$

then

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq a_1 G(x_n, x_{n+1}, x_{n+1}) + a_2 G(x_n, Tx_n, Tx_n) \\ &\quad + a_3 G(x_{n+1}, Tx_{n+1}, Tx_{n+1}) + a_4 G(x_n, Tx_{n+1}, Tx_{n+1}) \\ &\quad + a_5 G(x_{n+1}, Tx_n, Tx_n) \\ &= a_1 G(x_n, x_{n+1}, x_{n+1}) + a_2 G(x_n, x_{n+1}, x_{n+1}) \\ &\quad + a_3 G(x_{n+1}, x_{n+2}, x_{n+2}) + a_4 G(x_n, x_{n+2}, x_{n+2}) \\ &\quad + a_5 G(x_{n+1}, x_{n+1}, x_{n+1}) \\ G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq a_1 G(x_n, x_{n+1}, x_{n+1}) + a_2 G(x_n, x_{n+1}, x_{n+1}) \\ &\quad + a_3 G(x_{n+1}, x_{n+2}, x_{n+2}) + a_4 G(x_n, x_{n+2}, x_{n+2}) \end{aligned} \quad (6)$$

using the properties of G -Metric

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq a_1 G(x_n, x_{n+1}, x_{n+1}) + a_2 G(x_n, x_{n+1}, x_{n+1}) \\ &\quad + a_3 G(x_{n+1}, x_{n+2}, x_{n+2}) + a_4 G(x_n, x_{n+1}, x_{n+1}) \\ &\quad + a_4 G(x_{n+1}, x_{n+2}, x_{n+2}) \end{aligned} \quad (7)$$

which simplifies to

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \left(\frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} \right) G(x_n, x_{n+1}, x_{n+1}) \quad (8)$$

Let $k = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}$.

Since $\sum_{i=1}^5 a_i < 1$ and $a_i \leq a_{i+1}$ then $k < 1$.

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq k G(x_n, x_{n+1}, x_{n+1}) \quad (9)$$

repeating the process above, it follows that for all $n > 0$

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq a_1 G(x_{n-1}, x_n, x_n) + a_2 G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) \\ &\quad + a_3 G(x_n, Tx_n, Tx_n) + a_4 G(x_{n-1}, Tx_n, Tx_n) \\ &\quad + a_5 G(x_n, Tx_{n-1}, Tx_{n-1}) \\ &= a_1 G(x_{n-1}, x_n, x_n) + a_2 G(x_{n-1}, x_n, x_n) \\ &\quad + a_3 G(x_n, x_{n+1}, x_{n+1}) + a_4 G(x_{n-1}, x_{n+1}, x_{n+1}) \\ &\quad + a_5 G(x_n, x_n, x_n) \end{aligned} \quad (10)$$

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq a_1 G(x_{n-1}, x_n, x_n) + a_2 G(x_{n-1}, x_n, x_n) \\ &\quad + a_3 G(x_n, x_{n+1}, x_{n+1}) + a_4 G(x_{n-1}, x_{n+1}, x_{n+1}) \end{aligned} \quad (11)$$

Using property the properties of G-metrics

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq a_1 G(x_{n-1}, x_n, x_n) + a_2 G(x_{n-1}, x_n, x_n) \\ &\quad + a_3 G(x_n, x_{n+1}, x_{n+1}) + a_4 G(x_{n-1}, x_n, x_n) \\ &\quad + a_4 G(x_n, x_{n+1}, x_{n+1}) \end{aligned} \quad (12)$$

Simplifying further

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq \left(\frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} \right) G(x_{n-1}, x_n, x_n) \\ G(x_n, x_{n+1}, x_{n+1}) &\leq k G(x_{n-1}, x_n, x_n) \end{aligned} \quad (13)$$

Applying Lemma (4.3), $\{x_n\}$ is a Cauchy sequence. As (X, G) is complete, it is convergent, and there exists $x^* \in X$ such that $\{x_n\} \rightarrow x^*$. To show that x^* is a fixed point of T , consider

$$\begin{aligned}
G(x_n, Tx^*, Tx^*) &= G(Tx_{n-1}, Tx^*, Tx^*) \\
&\leq a_1 G(x_{n-1}, x^*, x^*) + a_2 G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) \\
&\quad + a_3 G(x^*, Tx^*, Tx^*) + a_4 G(x_{n-1}, Tx^*, Tx^*) \\
&\quad + a_5 G(x^*, Tx_{n-1}, Tx_{n-1}) \\
&= a_1 G(x_{n-1}, x^*, x^*) + a_2 G(x_{n-1}, x_n, x_n) + a_3 G(x^*, Tx^*, Tx^*) \\
&\quad + a_4 G(x_{n-1}, Tx^*, Tx^*) + a_5 G(x^*, x_n, x_n) \\
G(x_n, Tx^*, Tx^*) &\leq a_1 G(x_{n-1}, x^*, x^*) + a_2 G(x_{n-1}, x_n, x_n) + a_3 G(x^*, Tx^*, Tx^*) \\
&\quad + a_4 G(x_{n-1}, Tx^*, Tx^*) + a_5 G(x^*, x_n, x_n)
\end{aligned} \tag{14}$$

Taking limit as $n \rightarrow \infty$ and by the property that G is continuous,

$$\begin{aligned}
G(x^*, Tx^*, Tx^*) &\leq a_1 G(x^*, x^*, x^*) + a_2 G(x^*, x^*, x^*) + a_3 G(x^*, Tx^*, Tx^*) \\
&\quad + a_4 G(x^*, Tx^*, Tx^*) + a_5 G(x^*, x^*, x^*) \\
G(x^*, Tx^*, Tx^*) &\leq (a_3 + a_4) G(x^*, Tx^*, Tx^*)
\end{aligned} \tag{15}$$

since $a_3 + a_4 < 1$ the above inequality can only hold when $G(x^*, Tx^*, Tx^*) = 0$, hence $Tx^* = x^*$.

To show uniqueness, suppose $x^* \neq y^*$ such that $Ty^* = y^*$, then

$$\begin{aligned}
G(x^*, x^*, y^*) &= G(Tx^*, Tx^*, Ty^*) \\
&\leq a_1 G(x^*, x^*, y^*) + a_2 G(x^*, Tx^*, Tx^*) \\
&\quad + a_3 G(x^*, Tx^*, Tx^*) + a_4 G(x^*, Tx^*, Tx^*) + a_5 G(x^*, Tx^*, Tx^*) \\
&= a_1 G(x^*, x^*, y^*) + a_2 G(x^*, x^*, x^*) \\
&\quad + a_3 G(x^*, x^*, x^*) + a_4 G(x^*, x^*, x^*) + a_5 G(x^*, x^*, x^*) \\
G(x^*, x^*, y^*) &\leq a_1 G(x^*, x^*, y^*) < G(x^*, x^*, y^*)
\end{aligned} \tag{16}$$

This is a contradiction, hence $x^* = y^*$.

Proposition 3.1 *Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ a self mapping satisfying*

$$\begin{aligned}
G(Tx, Ty, Tz) &\leq a_1 G(x, y, z) + a_2 G(x, Tx, Tx) + a_3 G(y, Ty, Ty) \\
&\quad + a_4 G(x, Ty, Ty) + a_5 G(y, Tx, Tx)
\end{aligned} \tag{17}$$

for all $x, y, z \in X$ where each a_i are non negative constants satisfying

$$\sum_{i=1}^5 a_i < 1 \quad \text{and} \quad a_i \leq a_{i+1} \tag{18}$$

Let $x_0 \in X$, $\{x_n\} \subset X$ be defined by $x_{n+1} = Tx_n$. The sequence $\{x_n\}$ converges to a fixed point x^* of T . Let $\{y_n\}$ be an arbitrary sequence in X and set $\varepsilon_n = G(y_{n+1}, Ty_n, Ty_n)$

Then

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + (a_2 + a_3 + a_4 + a_5) \sum_{j=0}^n a_1^{n-j} G(x_j, Tx_j, Tx_j) \\
&\quad + a_1^{n+1} G(x_0, x_0, y_0) + \sum_{j=0}^n a_1^{n-j} \varepsilon_j
\end{aligned} \tag{19}$$

and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ if and only if $\lim_{n \rightarrow \infty} y_n = x^*$

Proof

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(y_{n+1}, x_{n+1}, x_{n+1}) + G(x_{n+1}, x^*, x^*) \\
&\leq G(x_{n+1}, x^*, x^*) + G(y_{n+1}, Ty_n, Ty_n) + G(Ty_n, x_{n+1}, x_{n+1}) \\
&= G(x_{n+1}, x^*, x^*) + G(Ty_n, Tx_n, Tx_n) + G(y_{n+1}, Ty_n, Ty_n) \\
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + G(Ty_n, Tx_n, Tx_n) + \varepsilon_n
\end{aligned}$$

Applying the contractive condition given above

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + G(Tx_n, Tx_n, Ty_n) + \varepsilon_n \\
&\leq G(x_{n+1}, x^*, x^*) + a_1 G(x_n, x_n, y_n) + a_2 G(x_n, Tx_n, Tx_n) \\
&\quad + a_3 G(x_n, Tx_n, Tx_n) + a_4 G(x_n, Tx_n, Tx_n) + a_5 G(x_n, Tx_n, Tx_n) + \varepsilon_n \\
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + (a_2 + a_3 + a_4 + a_5) G(x_n, Tx_n, Tx_n) \\
&\quad + a_1 G(x_n, x_n, y_n) + \varepsilon_n
\end{aligned} \tag{20}$$

Note that

$$\begin{aligned}
G(x_n, x_n, y_n) &= G(y_n, x_n, x_n) \\
&\leq G(y_n, Ty_{n-1}, Ty_{n-1}) + G(Ty_{n-1}, x_n, x_n) \\
&= G(Tx_{n-1}, Tx_{n-1}, Ty_{n-1}) + \varepsilon_{n-1} \\
&\leq a_1 G(x_{n-1}, x_{n-1}, y_{n-1}) + a_2 G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) \\
&\quad + a_3 G(x_{n-1}, Tx_{n-1}, T_{n-1}) + a_4 G(x_{n-1}, Tx_{n-1}, T_{n-1}) \\
&\quad + a_5 G(x_{n-1}, Tx_{n-1}, T_{n-1}) + \varepsilon_{n-1} \\
G(x_n, x_n, y_n) &\leq a_1 G(x_{n-1}, x_{n-1}, y_{n-1}) \\
&\quad + (a_2 + a_3 + a_4 + a_5) G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + \varepsilon_{n-1}
\end{aligned} \tag{21}$$

Then

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + (a_2 + a_3 + a_4 + a_5) G(x_n, Tx_n, Tx_n) \\
&\quad + a_1 (a_2 + a_3 + a_4 + a_5) G(x_{n-1}, Tx_{n-1}, x_{n-1}) \\
&\quad + a_1^2 G(x_{n-1}, x_{n-1}, y_{n-1}) + a_1 \varepsilon_{n-1} + \varepsilon_n
\end{aligned} \tag{22}$$

Repeating this process

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + (a_2 + a_3 + a_4 + a_5) G(x_n, Tx_n, Tx_n) \\
&\quad + a_1 (a_2 + a_3 + a_4 + a_5) G(x_{n-1}, Tx_{n-1}, x_{n-1}) \\
&\quad + a_1^2 (a_2 + a_3 + a_4 + a_5) G(x_{n-2}, Tx_{n-2}, x_{n-2}) \\
&\quad + a_1^3 G(x_{n-2}, x_{n-2}, y_{n-2}) + a_1^2 \varepsilon_{n-2} + a_1 \varepsilon_{n-1} + \varepsilon_n
\end{aligned} \tag{23}$$

Repeating the process n-times gives

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + (a_2 + a_3 + a_4 + a_5) \sum_{j=0}^n a_1^{n-j} G(x_j, Tx_j, Tx_j) \\
&\quad + a_1^{n+1} G(x_0, x_0, y_0) + \sum_{j=0}^n a_1^{n-j} \varepsilon_j
\end{aligned} \tag{24}$$

Suppose $\lim_{n \rightarrow \infty} y_n = x^*$

Then

$$\begin{aligned}
\varepsilon_n &= G(y_{n+1}, x^*, x^*) + G(x^*, Ty_n, Ty_n) \\
&\leq G(y_{n+1}, x^*, x^*) + 2G(x^*, x^*, Ty_n) \\
&\leq G(y_{n+1}, x^*, x^*) + 2a_1G(x^*, x^*, y_n) + 2a_2G(x^*, Tx^*, Tx^*) \\
&\quad + 2a_3G(x^*, Tx^*, Tx^*) + 2a_4G(x^*, Tx^*, Tx^*) + 2a_5G(x^*, Tx^*, Tx^*) \\
\varepsilon_n &\leq G(y_{n+1}, x^*, x^*) + 2a_1G(x^*, x^*, y_n)
\end{aligned} \tag{25}$$

Taking the limit as $n \rightarrow \infty$

Then

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0$$

Conversely, Suppose $\lim_{n \rightarrow \infty} \varepsilon_n = 0$

Consider the coefficients $\{a_1^{n-j}\}$

Set

$$A_{nj} = a_1^{n-j}$$

Then, $\lim_{n \rightarrow \infty} A_{nj} = 0$ for each j

and

$$\sum_{j=0}^n A_{nj} = \frac{1-a_1^{n+1}}{1-a_1} \rightarrow (1-a_1)^{-1} \text{ as } n \rightarrow \infty \tag{26}$$

Since $\lim_{n \rightarrow \infty} \varepsilon_n = 0$

Then,

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n a_1^{n-j} \varepsilon_j = 0$$

Since $\lim_{n \rightarrow \infty} G(x_{n+1}, x^*, x^*) = 0$

Then,

$$\lim_{n \rightarrow \infty} G(x_n, Tx_n, Tx_n) = 0$$

So that

$$\lim_{n \rightarrow \infty} (a_2 + a_3 + a_4 + a_5) \sum_{j=0}^n a_1^{n-j} G(x_j, Tx_j, Tx_j) = 0$$

Furthermore, $\lim_{n \rightarrow \infty} a_1^{n+1} G(x_0, x_0, y_0) = 0$ since $a_1 \in [0, 1)$

Then

$$\lim_{n \rightarrow \infty} G(y_{n+1}, x^*, x^*) = 0 \tag{27}$$

That is $\{y_n\}$ converges to the point x^* .

Corollary 3.1 Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying the condition

$$\begin{aligned}
G(Tx, Ty, Tz) &\leq a_1G(x, y, z) + a_2G(x, Tx, Tx) + a_3G(y, Ty, Ty) \\
&\quad + a_4G(x, Ty, Ty) + a_5G(y, Tx, Tx)
\end{aligned} \tag{28}$$

for all $x, y, z \in X$ where each a_i are nonnegative constants satisfying

$$\sum_{i=1}^5 a_i < 1 \quad \text{and} \quad a_i \leq a_{i+1} \quad (29)$$

Then

$$\begin{aligned} G(T^i x, T^i y, T^i z) &\leq a_1^i G(x, y, z) + a_2 \sum_{j=1}^i a_1^{j-1} G(T^{i-j} x, T^{i-j+1} x, T^{i-j+1} x) \\ &\quad + a_3 \sum_{j=1}^i a_1^{j-1} G(T^{i-j} y, T^{i-j+1} y, T^{i-j+1} y) \\ &\quad + a_4 \sum_{j=1}^i a_1^{j-1} G(T^{i-j} x, T^{i-j+1} y, T^{i-j+1} y) \\ &\quad + a_5 \sum_{j=1}^i a_1^{j-1} G(T^{i-j} y, T^{i-j+1} x, T^{i-j+1} x) \end{aligned} \quad (30)$$

Proof

$$\begin{aligned} G(T^i x, T^i y, T^i z) &\leq a_1 G(T^{i-1} x, T^{i-1} y, T^{i-1} z) + a_2 G(T^{i-1} x, T^i x, T^i x) \\ &\quad + a_3 G(T^{i-1} y, T^i y, T^i y) + a_4 G(T^{i-1} x, T^i y, T^i y) \\ &\quad + a_5 G(T^{i-1} y, T^i x, T^i x) \end{aligned} \quad (31)$$

repeating the process above

$$\begin{aligned} G(T^i x, T^i y, T^i z) &\leq a_1^2 G(T^{i-2} x, T^{i-2} y, T^{i-2} z) + a_2 G(T^{i-1} x, T^i x, T^i x) \\ &\quad + a_1 a_2 G(T^{i-2} x, T^{i-1} x, T^{i-1} x) + a_3 G(T^{i-1} y, T^i y, T^i y) \\ &\quad + a_1 a_3 G(T^{i-2} y, T^{i-1} y, T^{i-1} y) + a_4 G(T^{i-1} x, T^i y, T^i y) \\ &\quad + a_1 a_4 G(T^{i-2} x, T^{i-1} y, T^{i-1} y) + a_5 G(T^{i-1} y, T^i x, T^i x) \\ &\quad + a_1 a_5 G(T^{i-2} y, T^{i-1} x, T^{i-1} x) \end{aligned} \quad (32)$$

repeating the process again

$$\begin{aligned} G(T^i x, T^i y, T^i z) &\leq a_1^3 G(T^{i-3} x, T^{i-3} y, T^{i-3} z) + a_2 G(T^{i-1} x, T^i x, T^i x) \\ &\quad + a_1 a_2 G(T^{i-2} x, T^{i-1} x, T^{i-1} x) + a_1^2 a_2 G(T^{i-3} x, T^{i-2} x, T^{i-2} x) \\ &\quad + a_3 G(T^{i-1} y, T^i y, T^i y) + a_1 a_3 G(T^{i-2} y, T^{i-1} y, T^{i-1} y) \\ &\quad + a_1^2 a_3 G(T^{i-3} y, T^{i-2} y, T^{i-2} y) + a_4 G(T^{i-1} x, T^i y, T^i y) \\ &\quad + a_1 a_4 G(T^{i-2} x, T^{i-1} y, T^{i-1} y) + a_1^2 a_4 G(T^{i-3} x, T^{i-2} y, T^{i-2} y) \\ &\quad + a_5 G(T^{i-1} y, T^i x, T^i x) + a_1 a_5 G(T^{i-2} y, T^{i-1} x, T^{i-1} x) \\ &\quad + a_1^2 a_5 G(T^{i-3} y, T^{i-2} x, T^{i-2} x) \end{aligned} \quad (33)$$

repeating the process i-times will give

$$\begin{aligned}
G(T^i x, T^i y, T^i z) &\leq a_1^i G(x, y, z) + a_2 \sum_{j=1}^i a_1^{j-1} G(T^{i-j} x, T^{i-j+1} x, T^{i-j+1} x) \\
&\quad + a_3 \sum_{j=1}^i a_1^{j-1} G(T^{i-j} y, T^{i-j+1} y, T^{i-j+1} y) \\
&\quad + a_4 \sum_{j=1}^i a_1^{j-1} G(T^{i-j} x, T^{i-j+1} y, T^{i-j+1} y) \\
&\quad + a_5 \sum_{j=1}^i a_1^{j-1} G(T^{i-j} y, T^{i-j+1} x, T^{i-j+1} x)
\end{aligned} \tag{34}$$

Theorem 3.2 Let (X, G) be a complete G -metric space, and let a, b, c be nonnegative functions satisfying

$$\text{Sup}\{a(x, y, z) + b(x, y, z) + c(x, y, z)\} \leq \lambda < 1 \tag{35}$$

$x, y \in X$ such that, for each $x, y, z \in X$. Let $T : X \rightarrow X$ be a mapping satisfying the contractive condition.

$$\begin{aligned}
G(Tx, Ty, Tz) &\leq a(x, y, z)G(x, Tx, Tx) + bG(x, y, z)G(y, Ty, Ty) \\
&\quad + cG(x, y, z)G(x, y, z)
\end{aligned} \tag{36}$$

for all $x, y, z \in X$ then T has a unique fixed point x^* , i.e $Tx^* = x^*$.

Proof

Let x_0 be an arbitrary point in X . Let $\{x_n\}$ be a sequence in X defined by

$$x_{n+1} = Tx_n \tag{37}$$

if there is some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is a fixed point of T . Assume

$x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.

Let $x = x_n$, and $y = z = x_{n+1}$ then

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\
&\leq a(G(x_n, x_{n+1}, x_{n+1}))G(x_n, Tx_n, Tx_n) \\
&\quad + b(G(x_n, x_{n+1}, x_{n+1}))G(x_{n+1}, Tx_{n+1}, Tx_{n+1}) \\
&\quad + c(G(x_n, x_{n+1}, x_{n+1}))G(x_n, x_{n+1}, x_{n+1}) \\
&= a(G(x_n, x_{n+1}, x_{n+1}))G(x_n, x_{n+1}, x_{n+1}) \\
&\quad + b(G(x_n, x_{n+1}, x_{n+1}))G(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad + c(G(x_n, x_{n+1}, x_{n+1}))G(x_n, x_{n+1}, x_{n+1})
\end{aligned} \tag{38}$$

simplifying further

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \frac{[a(G(x_n, x_{n+1}, x_{n+1})) + c(G(x_n, x_{n+1}, x_{n+1}))]}{1 - b(G(x_n, x_{n+1}, x_{n+1}))} G(x_n, x_{n+1}, x_{n+1}) \tag{39}$$

Choose λ such that

$$\frac{a(t) + c(t)}{1 - b(t)} \leq \lambda < 1 \tag{40}$$

Then

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda G(x_n, x_{n+1}, x_{n+1}) \tag{41}$$

By Lemma (3.1.3), $\{x_n\}$ is a Cauchy sequence, since (X, G) is complete $\{x_n\}$ converges to a point $x^* \in X$.

Suppose

$$Tx^* \neq x^*$$

Then

$$G(x^*, Tx^*, Tx^*) > 0$$

$$\begin{aligned} G(x_n, Tx^*, Tx^*) &= G(Tx_{n-1}, Tx^*, Tx^*) \\ &\leq a(G(x_{n-1}, x^*, x^*))G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) \\ &\quad + b(G(x_{n-1}, x^*, x^*))G(x^*, Tx^*, Tx^*) \\ &\quad + c(G(x_{n-1}, x^*, x^*))(G(x_{n-1}, x^*, x^*)) \end{aligned} \quad (42)$$

$$\begin{aligned} G(x_n, Tx^*, Tx^*) &\leq a(G(x_{n-1}, x^*, x^*))G(x_{n-1}, x_n, x_n) + b(G(x_{n-1}, x^*, x^*))G(x^*, Tx^*, Tx^*) \\ &\quad + c(G(x_{n-1}, x^*, x^*))(G(x_{n-1}, x^*, x^*)) \end{aligned} \quad (43)$$

Taking the limit as $n \rightarrow \infty$

$$\begin{aligned} G(x^*, Tx^*, Tx^*) &\leq b(G(x^*, x^*, x^*))G(x^*, Tx^*, Tx^*) \\ G(x^*, Tx^*, Tx^*) &= 0 \\ Tx^* &= x^* \end{aligned} \quad (44)$$

Suppose $x^* \neq y^*$ then

$$\begin{aligned} G(x^*, y^*, y^*) &= G(Tx^*, Ty^*, Ty^*) \\ &\leq a(G(x^*, y^*, y^*))G(x^*, Tx^*, Tx^*) + b(G(x^*, y^*, y^*))G(y^*, Ty^*, Ty^*) \\ &\quad + c(G(x^*, y^*, y^*))G(x^*, y^*, y^*) \\ &\leq a(G(x^*, y^*, y^*))G(x^*, x^*, x^*) + b(G(x^*, y^*, y^*))G(y^*, y^*, y^*) \\ &\quad + c(G(x^*, y^*, y^*))G(x^*, y^*, y^*) \end{aligned} \quad (45)$$

$$G(x^*, y^*, y^*) \leq c(G(x^*, y^*, y^*))G(x^*, y^*, y^*) < G(x^*, y^*, y^*) < 0 \quad (46)$$

This is contradiction, hence $x^* = y^*$

Proposition 3.2 Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ a self mapping satisfying

$$G(Tx, Ty, Tz) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(x, y, z) \quad (47)$$

for all $x, y, z \in X$, $0 \leq a + b + c \leq 1$. Let $x_0 \in X$, $\{x_n\} \subset X$ be defined by $x_{n+1} = Tx_n$. The sequence $\{x_n\}$ converges to a fixed point x^* of T . Let $\{y_n\}$ be an arbitrary sequence in X and set $\varepsilon_n = G(y_{n+1}, Ty_n, Ty_n)$

Then

$$\begin{aligned} G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + (a+b) \sum_{j=0}^n c^{n-j} G(x_j, Tx_j, Tx_j) \\ &\quad + c^{n+1} G(x_0, x_0, y_0) + \sum_{j=0}^n c^{n-j} \varepsilon_j \end{aligned} \quad (48)$$

and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ if and only if $\lim_{n \rightarrow \infty} y_n = x^*$

Proof

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(y_{n+1}, x_{n+1}, x_{n+1}) + G(x_{n+1}, x^*, x^*) \\
&\leq G(x_{n+1}, x^*, x^*) + G(y_{n+1}, Ty_n, Ty_n) + G(Ty_n, x_{n+1}, x_{n+1}) \\
&= G(x_{n+1}, x^*, x^*) + G(Ty_n, Tx_n, Tx_n) + G(y_{n+1}, Ty_n, Ty_n)
\end{aligned}$$

$$G(y_{n+1}, x^*, x^*) \leq G(x_{n+1}, x^*, x^*) + G(Ty_n, Tx_n, Tx_n) + \varepsilon_n$$

Applying the contractive condition given above

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + G(Tx_n, Tx_n, Ty_n) + \varepsilon_n \\
&\leq G(x_{n+1}, x^*, x^*) + aG(x_n, Tx_n, Tx_n) + bG(x_n, Tx_n, Tx_n) \\
&\quad + cG(x_n, x_n, y_n) + \varepsilon_n \\
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + (a+b)G(x_n, Tx_n, Tx_n) \\
&\quad + cG(x_n, x_n, y_n) + \varepsilon_n
\end{aligned} \tag{49}$$

Note that

$$\begin{aligned}
G(x_n, x_n, y_n) &= G(y_n, x_n, x_n) \\
&\leq G(y_n, Ty_{n-1}, Ty_{n-1}) + G(Ty_{n-1}, x_n, x_n) \\
&= G(Tx_{n-1}, Tx_{n-1}, Ty_{n-1}) + \varepsilon_{n-1} \\
&\leq aG(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + bG(x_{n-1}, Tx_{n-1}, Tx_{n-1}) \\
&\quad + cG(x_{n-1}, x_{n-1}, y_{n-1}) + \varepsilon_{n-1} \\
G(x_n, x_n, y_n) &\leq (a+b)G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) \\
&\quad + cG(x_{n-1}, x_{n-1}, y_{n-1}) + \varepsilon_{n-1}
\end{aligned}$$

Then the inequality reduces to

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + (a+b)G(x_n, Tx_n, Tx_n) \\
&\quad + c[(a+b)G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + cG(x_{n-1}, x_{n-1}, y_{n-1}) + \varepsilon_{n-1}] + \varepsilon_n \\
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + (a+b)G(x_n, Tx_n, Tx_n) \\
&\quad + c(a+b)G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + c^2G(x_{n-1}, x_{n-1}, y_{n-1}) \\
&\quad + c\varepsilon_{n-1} + \varepsilon_n
\end{aligned} \tag{50}$$

repeating this process

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + (a+b)G(x_n, Tx_n, Tx_n) \\
&\quad + c(a+b)G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + c^2(a+b)G(x_{n-2}, Tx_{n-2}, Tx_{n-2}) \\
&\quad + c^3G(x_{n-2}, x_{n-2}, y_{n-2}) + c^2\varepsilon_{n-2} + c\varepsilon_{n-1} + \varepsilon_n
\end{aligned} \tag{51}$$

repeating the process n-times gives

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + (a+b) \sum_{j=0}^n c^{n-j} G(x_j, Tx_j, Tx_j) \\
&\quad + c^{n+1} G(x_0, x_0, y_0) + \sum_{j=0}^n c^{n-j} \varepsilon_j
\end{aligned} \tag{52}$$

Suppose $\lim_{n \rightarrow \infty} y_n = x^*$

then

$$\begin{aligned}
\varepsilon_n &= G(y_{n+1}, x^*, x^*) + G(x^*, Ty_n, Ty_n) \\
&\leq G(y_{n+1}, x^*, x^*) + 2G(x^*, x^*, Ty_n) \\
&\leq G(y_{n+1}, x^*, x^*) + 2aG(x^*, Tx^*, Tx^*) + 2bG(x^*, Tx^*, Tx^*) \\
&\quad + 2cG(x^*, x^*, y_n) \\
\varepsilon_n &\leq G(y_{n+1}, x^*, x^*) + 2cG(x^*, x^*, y_n)
\end{aligned} \tag{53}$$

Taking the limit as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0$$

Conversely, Suppose $\lim_{n \rightarrow \infty} \varepsilon_n = 0$

Consider the coefficients $\{c^{n-j}\}$

set

$$A_{nj} = c^{n-j}$$

then $\lim_{n \rightarrow \infty} A_{nj} = 0$ for each j

and

$$\sum_{j=0}^n A_{nj} = \frac{1-c^{n+1}}{1-c} \rightarrow (1-c)^{-1} \text{ as } n \rightarrow \infty \tag{54}$$

since $\lim_{n \rightarrow \infty} \varepsilon_n = 0$

then

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n c^{n-j} \varepsilon_j = 0$$

since $\lim_{n \rightarrow \infty} G(x_{n+1}, x^*, x^*) = 0$

then

$$\lim_{n \rightarrow \infty} G(x_n, Tx_n, Tx_n) = 0$$

so that

$$\lim_{n \rightarrow \infty} (a+b) \sum_{j=0}^n c^{n-j} G(x_j, Tx_j, Tx_j) = 0$$

and

$$\lim_{n \rightarrow \infty} c^{n+1} G(x_0, x_0, y_0) = 0 \text{ since } c \in [0,1)$$

then

$$\lim_{n \rightarrow \infty} G(y_{n+1}, x^*, x^*) = 0 \tag{55}$$

That is $\{y_n\}$ converges to the point x^* .

Corollary 3.2 Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying the condition

$$G(Tx, Ty, Tz) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(x, y, z) \tag{56}$$

for all $x, y, z \in X$, $0 \leq a+b+c \leq 1$

Then

$$\begin{aligned}
G(T^i x, T^i y, T^i z) &\leq a \sum_{j=1}^i c^{j-1} \left(\frac{a+c}{1-b} \right)^{i-j} G(x, Tx, Tx) \\
&\quad + b \sum_{j=1}^i c^{j-1} \left(\frac{a+c}{1-b} \right)^{i-j} G(y, Ty, Ty) + c^i G(x, y, z)
\end{aligned} \tag{57}$$

Theorem 3.3 Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying any of the following conditions

$$G(Tx, Ty, Tz) \leq a\{G(x, Ty, Ty) + G(y, Tx, Tx)\} \quad (58)$$

or

$$G(Tx, Ty, Tz) \leq a\{G(x, Tx, Tx) + G(y, Ty, Ty)\} \quad (59)$$

where a is a constant satisfying $0 < a < \frac{1}{2}$, and $x, y \in X$. Then T has a fixed point in

$$X \text{ (say } x^*), Tx^* = x^*$$

Proof

Let $x_0 \in X$ be an arbitrary point. Define the sequence $\{x_n\} \in X$ by

$$x_{n+1} = Tx_n \text{ for all } n \in N. \quad (60)$$

Set $x = x_n$ and $y = z = x_{n+1}$,

using the first contractive condition above

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq a\{G(x_n, Tx_{n+1}, Tx_{n+1}) + G(x_{n+1}, Tx_n, Tx_n)\} \\ &= a\{G(x_n, x_{n+2}, x_{n+2}) + G(x_{n+1}, x_{n+1}, x_{n+1})\} \\ &= aG(x_n, x_{n+2}, x_{n+2}) \\ &\leq aG(x_n, x_{n+1}, x_{n+1}) + aG(x_{n+1}, x_{n+2}, x_{n+2}) \end{aligned} \quad (61)$$

simplifying further,

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \frac{a}{1-a} G(x_n, x_{n+1}, x_{n+1}) \quad (62)$$

similarly, using the second contractive condition

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq a\{G(x_n, Tx_n, Tx_n) + G(x_{n+1}, Tx_{n+1}, Tx_{n+1})\} \\ &= a\{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})\} \end{aligned} \quad (63)$$

simplifying further gives

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \frac{a}{1-a} G(x_n, x_{n+1}, x_{n+1}) \quad (64)$$

Applying Lemma (3.1.3), $\{x_n\}$ is a Cauchy sequence. As (X, G) is complete, it is convergent, so there exists $x^* \in X$ such that $\{x_n\} \rightarrow x^*$.

To show that x^* is a fixed point of T , consider the following

$$\begin{aligned} G(x_n, Tx^*, Tx^*) &= G(Tx_{n-1}, Tx^*, Tx^*) \\ &\leq a\{G(x_{n-1}, Tx^*, Tx^*) + G(x^*, Tx_{n-1}, Tx_{n-1})\} \\ &= aG(x_{n-1}, Tx^*, Tx^*) + aG(x^*, Tx_{n-1}, Tx_{n-1}) \\ Gx_n, Tx^*, Tx^*) &\leq aG(x_{n-1}, Tx^*, Tx^*) + aG(x^*, x_n, x_n) \end{aligned} \quad (65)$$

Taking the limit as $n \rightarrow \infty$

$$G(x^*, Tx^*, Tx^*) \leq aG(x^*, x^*, x^*) + bG(x^*, x^*, x^*) = 0$$

hence, $Tx^* = x^*$.

Suppose $x^* \neq y^*$ such that $Ty^* = y^*$. Then $G(x^*, x^*, y^*) > 0$ and

$$\begin{aligned}
G(x^*, x^*, y^*) &= G(Tx^*, Tx^*, Ty^*) \\
&\leq \alpha G(x^*, Tx^*, Tx^*) + bG(x^*, Tx^*, Tx^*) \\
G(x^*, x^*, y^*) &= 0
\end{aligned} \tag{66}$$

Hence $x^* = y^*$.

Proposition 3.3 Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ a self mapping satisfying

$$G(Tx, Ty, Tz) \leq \alpha G(x, Tx, Tx) + \psi G(x, y, z) \tag{67}$$

for all $x, y, z \in X$, $0 < \alpha + \psi \leq 1$. Let $x_0 \in X$, $\{x_n\} \subset X$ be defined by $x_{n+1} = Tx_n$. The sequence $\{x_n\}$ converges to a fixed point x^* of T . Let $\{y_n\}$ be an arbitrary sequence in X and set $\varepsilon_n = G(y_{n+1}, Ty_n, Ty_n)$

Then

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + \alpha \sum_{j=0}^n \psi^{n-j} G(x_j, Tx_j, Tx_j) \\
&\quad + \psi^{n+1} G(x_0, x_0, y_0) + \sum_{j=0}^n \psi^{n-j} \varepsilon_j
\end{aligned} \tag{68}$$

and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ if and only if $\lim_{n \rightarrow \infty} y_n = x^*$

Proof

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(y_{n+1}, x_{n+1}, x_{n+1}) + G(x_{n+1}, x^*, x^*) \\
&\leq G(x_{n+1}, x^*, x^*) + G(y_{n+1}, Ty_n, Ty_n) + G(Ty_n, x_{n+1}, x_{n+1}) \\
&= G(x_{n+1}, x^*, x^*) + G(Ty_n, Tx_n, Tx_n) + G(y_{n+1}, Ty_n, Ty_n) \\
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + G(Ty_n, Tx_n, Tx_n) + \varepsilon_n
\end{aligned}$$

Applying the contractive condition given above

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + G(Tx_n, Tx_n, Ty_n) + \varepsilon_n \\
&\leq G(x_{n+1}, x^*, x^*) + \alpha G(x_n, Tx_n, Tx_n) \\
&\quad + \psi G(x_n, x_n, y_n) + \varepsilon_n \\
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + \alpha G(x_n, Tx_n, Tx_n) \\
&\quad + \alpha \psi G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + \psi^2 G(x_{n-1}, x_{n-1}, y_{n-1}) \\
&\quad + \psi \varepsilon_{n-1} + \varepsilon_n
\end{aligned}$$

repeating this process n-times leads to

$$\begin{aligned}
G(y_{n+1}, x^*, x^*) &\leq G(x_{n+1}, x^*, x^*) + \alpha \sum_{j=0}^n \psi^{n-j} G(x_j, Tx_j, Tx_j) \\
&\quad + \psi^{n+1} G(x_0, x_0, y_0) + \sum_{j=0}^n \psi^{n-j} \varepsilon_j
\end{aligned} \tag{69}$$

Suppose $\lim_{n \rightarrow \infty} y_n = x^*$

then

$$\begin{aligned}
\varepsilon_n &= G(y_{n+1}, Ty_n, Ty_n) \\
&\leq G(y_{n+1}, x^*, x^*) + G(x^*, Ty_n, Ty_n) \\
\varepsilon_n &\leq G(y_{n+1}, x^*, x^*) + G(x^*, Tx^*, Tx^*) + G(x^*, y_n, y_n)
\end{aligned} \tag{70}$$

Taking the limit as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0$$

Conversely

Suppose $\lim_{n \rightarrow \infty} \varepsilon_n = 0$

consider the coefficients $\{\psi^{n-j}\}$

set

$$A_{nj} = \psi^{n-j}$$

then $\lim_{n \rightarrow \infty} A_{nj} = 0$ for each j

$$\sum_{j=0}^n A_{nj} = \frac{1-\psi^{n+1}}{1-\psi} \rightarrow (1-\psi)^{-1} \text{ as } n \rightarrow \infty \quad (71)$$

since $\lim_{n \rightarrow \infty} \varepsilon_n = 0$

then

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n \psi^{n-j} \varepsilon_j = 0$$

since $\lim_{n \rightarrow \infty} G(x_{n+1}, x^*, x^*) = 0$

then

$$\lim_{n \rightarrow \infty} G(x_n, Tx_n, Tx_n) = 0$$

so that

$$\lim_{n \rightarrow \infty} \alpha \sum_{j=0}^n \psi^{n-j} G(x_j, Tx_j, Tx_j) = 0$$

furthermore $\lim_{n \rightarrow \infty} \psi^{n+1} G(x_0, x_0, y_0) = 0$ since $\psi \in [0,1)$

then

$$\lim_{n \rightarrow \infty} G(y_{n+1}, x^*, x^*) = 0 \quad (72)$$

that is $\{y_n\}$ converges to the point x^* .

Corollary 3.3 Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a self mapping satisfying

$$G(Tx, Ty, Tz) \leq \alpha G(x, Tx, Tx) + \psi G(x, y, z) \quad (73)$$

for all $x, y, z \in X$ where $0 \leq \alpha + \psi \leq 1$

Then

$$G(T^i x, T^i y, T^i z) \leq \alpha \sum_{j=1}^i \psi^{j-1} G(T^{i-j} x, T^{i-j+1} x, T^{i-j+1} x) + \psi^i G(x, y, z) \quad (74)$$

Theorem 3.4 Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying the contractive condition,

$$G(Tx, Ty, Tz) \leq aG(x, Ty, Ty) + bG(y, Tx, Tx) + cG(x, y, z) \quad (75)$$

where a, b, c are nonnegative numbers satisfying $a + b + c < 1$ and $a \leq b \leq c$, then T has a unique fixed point in X .

Proof

The proof can be sourced from (Rauf *et al.*, 2017).

Theorem 3.5 Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying the contractive condition,

$$G(Tx, Ty, Tz) \leq \lambda \max\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\} \quad (76)$$

where $\lambda \in [0,1)$, $x, y, z \in X$. Then T has a unique fixed point.

Proof

Let x_0 be an arbitrary point in X . Let $\{x_n\}$ be a sequence in X defined by $x_n = Tx_{n-1}$.

Let $x = x_n$ and $y = z = x_{n+1}$.

Then

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq \lambda \max\{G(x_n, Tx_n, Tx_n), G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \\ &\quad G(x_{n+1}, Tx_{n+1}, Tx_{n+1})\} \\ G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \lambda \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\} \end{aligned} \quad (77)$$

There are two cases to examine

Case 1: When

$$\max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\} = G(x_{n+1}, x_{n+2}, x_{n+2})$$

then

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda G(x_{n+1}, x_{n+2}, x_{n+2}) \quad (78)$$

which is not possible since $\lambda < 1$.

Case 2: When

$$\max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\} = G(x_n, x_{n+1}, x_{n+1})$$

then

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda G(x_n, x_{n+1}, x_{n+1}) \quad (79)$$

By Lemma (3.1.3) $\{x_n\}$ is a Cauchy sequence. Since (X, G) is complete, then $\{x_n\}$ converges

to a point x^* in X . To show that x^* is a fixed point of T , consider

$$\begin{aligned} G(x_n, Tx^*, Tx^*) &= G(Tx_{n-1}, Tx^*, Tx^*) \\ &\leq \lambda \max\{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x^*, Tx^*, Tx^*), G(x^*, Tx^*, Tx^*)\} \\ G(x_n, Tx^*, Tx^*) &\leq \lambda \max\{G(x_{n-1}, x_n, x_n), G(x^*, Tx^*, Tx^*)\} \end{aligned}$$

since G is jointly continuous in three of its variables, taking the limit as $n \rightarrow \infty$

$$G(x^*, Tx^*, Tx^*) \leq \lambda G(x^*, Tx^*, Tx^*) \quad (80)$$

As $\lambda < 1$ this inequality can only hold when $G(x^*, Tx^*, Tx^*) = 0$. This implies that $Tx^* = x^*$.

To show uniqueness, suppose $x^* \neq y^*$ such that $Ty^* = y^*$, then

$$\begin{aligned} G(x^*, y^*, y^*) &= G(Tx^*, Ty^*, Ty^*) \\ &\leq \lambda \max\{G(x^*, Tx^*, Tx^*), G(y^*, Ty^*, Ty^*)\} = 0 \end{aligned} \quad (81)$$

Hence $x^* = y^*$.

Theorem 3.6 Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying the following condition

$$\begin{aligned} G(Tx, Ty, Tz) &\leq \lambda \max\{G(x, y, z), \\ &\quad \frac{G(x, Tx, Tx) + G(y, Ty, Ty)}{2}, \frac{G(x, Ty, Ty) + G(y, Tx, Tx)}{2}\} \end{aligned} \quad (82)$$

for all $x, y, z \in X$ and $\lambda \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Proof

Let x_0 be an arbitrary point in X . Let $\{x_n\}$ be a sequence in X defined by $x_n = Tx_{n-1}$.

Let $x = x_n$ and $y = z = x_{n+1}$

then

$$\begin{aligned}
G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\
&\leq \lambda \max \left\{ G(x_n, x_{n+1}, x_{n+1}), \right. \\
&\quad \left. \frac{G(x_n, Tx_n, Tx_n) + G(x_{n+1}, Tx_{n+1}, Tx_{n+1})}{2}, \right. \\
&\quad \left. \frac{G(x_n, Tx_{n+1}, Tx_{n+1}) + G(x_{n+1}, Tx_n, Tx_n)}{2} \right\} \\
&= \lambda \max \left\{ G(x_n, x_{n+1}, x_{n+1}), \right. \\
&\quad \left. \frac{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})}{2}, \right. \\
&\quad \left. \frac{G(x_n, x_{n+2}, x_{n+2}) + G(x_{n+1}, x_{n+1}, x_{n+1})}{2} \right\} \\
G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \lambda \max \left\{ G(x_n, x_{n+1}, x_{n+1}), \right. \\
&\quad \left. \frac{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})}{2}, \right. \\
&\quad \left. \frac{G(x_n, x_{n+2}, x_{n+2})}{2} \right\} \\
G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \lambda M
\end{aligned} \tag{83}$$

where

$$\begin{aligned}
M &= \max \left\{ G(x_n, x_{n+1}, x_{n+1}), \right. \\
&\quad \left. \frac{[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})]}{2}, \right. \\
&\quad \left. \frac{G(x_n, x_{n+2}, x_{n+2})}{2} \right\}
\end{aligned} \tag{84}$$

There are three cases to consider

Case 1: When $M = G(x_n, x_{n+1}, x_{n+1})$

then

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda G(x_n, x_{n+1}, x_{n+1}) \tag{85}$$

Case 2: When $M = \frac{[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})]}{2}$

then

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda \frac{[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})]}{2} \tag{86}$$

which gives

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \frac{\lambda}{2 - \lambda} G(x_n, x_{n+1}, x_{n+1}) \tag{87}$$

Case 3: When

$$M = \frac{G(x_n, x_{n+2}, x_{n+2})}{2} \tag{88}$$

then

$$\begin{aligned}
G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \frac{\lambda}{2} G(x_n, x_{n+2}, x_{n+2}) \\
&\leq \frac{\lambda}{2} [G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})]
\end{aligned} \tag{89}$$

simplifying the above

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \frac{\lambda}{2-\lambda} G(x_n, x_{n+1}, x_{n+1}) \quad (90)$$

since

$$\lambda \geq \frac{\lambda}{2-\lambda} \quad (91)$$

from the three cases

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda G(x_n, x_{n+1}, x_{n+1}) \quad (92)$$

By Lemma (3.1.3) $\{x_n\}$ is a Cauchy sequence. Since (X, G) is complete, $\{x_n\}$ converges to a point $x^* \in X$. To show that x^* is a fixed point of T

$$\begin{aligned} G(x_n, Tx^*, Tx^*) &= G(Tx_{n-1}, Tx^*, Tx^*) \\ &\leq \lambda \max\{G(x_{n-1}, x^*, x^*), \\ &\quad \frac{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(x^*, Tx^*, Tx^*)}{2}, \\ &\quad \frac{G(x_{n-1}, Tx^*, Tx^*) + G(x^*, Tx_{n-1}, Tx_{n-1})}{2}\} \\ G(x_n, Tx^*, Tx^*) &\leq \lambda \max\{G(x_{n-1}, x^*, x^*), \\ &\quad \frac{G(x_{n-1}, x_n, x_n) + G(x^*, Tx^*, Tx^*)}{2}, \\ &\quad \frac{G(x_{n-1}, Tx^*, Tx^*) + G(x^*, x_n, x_n)}{2}\} \end{aligned} \quad (93)$$

Since G is jointly continuous in all of its variables, taking the limit as $n \rightarrow \infty$

$$\begin{aligned} G(x^*, Tx^*, Tx^*) &\leq \lambda \max\{G(x^*, x^*, x^*), \frac{[G(x^*, x^*, x^*) + G(x^*, Tx^*, Tx^*)]}{2}, \\ &\quad \frac{G(x^*, Tx^*, Tx^*) + G(x^*, x^*, x^*)}{2}\} \\ G(x^*, Tx^*, Tx^*) &\leq \frac{\lambda}{2} G(x^*, Tx^*, Tx^*) \end{aligned} \quad (94)$$

Since $\lambda \in [0, \frac{1}{2})$, the inequality can only hold when $Tx^* = x^*$.

To show uniqueness, let $y^* \in X$ such that $Ty^* = y^*$ and $x^* \neq y^*$

Then $G(x^*, y^*, y^*) > 0$

$$\begin{aligned} G(x^*, y^*, y^*) &= G(Tx^*, Ty^*, Ty^*) \\ &\leq \lambda \max\{G(x^*, y^*, y^*), \frac{[G(x^*, Tx^*, Tx^*) + G(y^*, Ty^*, Ty^*)]}{2}, \\ &\quad \frac{[G(x^*, Ty^*, Ty^*) + G(y^*, Tx^*, Tx^*)]}{2}\} \\ &\leq \lambda \max\{G(x^*, y^*, y^*), \frac{[G(x^*, y^*, y^*) + G(y^*, x^*, x^*)]}{2}\} \end{aligned} \quad (95)$$

There are two cases to consider

Case 1: When

$$\max\{G(x^*, y^*, y^*), \frac{[G(x^*, y^*, y^*) + G(y^*, x^*, x^*)]}{2}\} = G(x^*, y^*, y^*) \quad (96)$$

since $\lambda < \frac{1}{2}$

$$G(x^*, y^*, y^*) \leq \lambda G(x^*, y^*, y^*) < G(x^*, y^*, y^*) \quad (97)$$

this is a contradiction hence $x^* = y^*$.

Case 2: When

$$\max\{G(x^*, y^*, y^*), \frac{[G(x^*, y^*, y^*) + G(y^*, x^*, x^*)]}{2}\} = \frac{[G(x^*, y^*, y^*) + G(y^*, x^*, x^*)]}{2} \quad (98)$$

then

$$\begin{aligned} G(x^*, y^*, y^*) &\leq \lambda \frac{[G(x^*, y^*, y^*) + G(y^*, x^*, x^*)]}{2} \\ &\leq \frac{\lambda}{2} [G(x^*, y^*, y^*) + 2G(x^*, y^*, y^*)] \\ &= \frac{3\lambda}{2} G(x^*, y^*, y^*) \\ G(x^*, y^*, y^*) &\leq \frac{3\lambda}{2} G(x^*, y^*, y^*) < G(x^*, y^*, y^*) \end{aligned} \quad (99)$$

This is also a contradiction, hence $x^* = y^*$.

Theorem 3.7 Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying the following condition,

$$G(Tx, Ty, Tz) \leq \lambda \max\{G(x, Ty, Ty), G(y, Tx, Tx)\} \quad (100)$$

where $\lambda \in [0, 1)$, $x, y, z \in X$. Then T has a unique fixed point in X .

Proof

Let x_0 be an arbitrary point in X . Let $\{x_n\}$ be a sequence in X defined by $x_n = Tx_{n-1}$.

Let $x = x_n$ and $y = z = x_{n+1}$.

Then

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq \lambda \max\{G(x_n, Tx_{n+1}, Tx_{n+1}), G(x_{n+1}, Tx_n, Tx_n)\} \\ &\leq \lambda G(x_n, x_{n+2}, x_{n+2}) \\ G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \lambda G(x_n, x_{n+1}, x_{n+1}) + \lambda G(x_{n+1}, x_{n+2}, x_{n+2}) \\ G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \frac{\lambda}{1-\lambda} G(x_n, x_{n+1}, x_{n+1}) \end{aligned} \quad (101)$$

By Lemma (3.1.3), $\{x_n\}$ is a Cauchy sequence. Since (X, G) is complete, then $\{x_n\}$ converges to a point x^* in X . To show that x^* is a fixed point of T , consider

$$\begin{aligned} G(x_n, Tx^*, Tx^*) &= G(Tx_{n-1}, Tx^*, Tx^*) \\ &\leq \lambda \max\{G(x_{n-1}, Tx^*, Tx^*), G(x^*, Tx_{n-1}, Tx_{n-1})\} \\ G(x_n, Tx^*, Tx^*) &\leq \lambda \max\{G(x_{n-1}, x^*, x^*), G(x^*, x_n, x_n)\} \end{aligned}$$

Since G is jointly continuous in three of its variables, taking the limit as $n \rightarrow \infty$

$$G(x^*, Tx^*, Tx^*) \leq \lambda \max\{G(x^*, x^*, x^*), G(x^*, x^*, x^*)\} = 0 \quad (102)$$

This inequality can only hold when $G(x^*, Tx^*, Tx^*) = 0$. This implies that $Tx^* = x^*$.

To show uniqueness, suppose $y^* \neq x^*$ such that $Ty^* = y^*$, then

$$\begin{aligned} G(x^*, x^*, y^*) &= G(Tx^*, Tx^*, Ty^*) \\ &\leq \lambda \max\{G(x^*, Tx^*, Tx^*), G(x^*, Tx^*, Tx^*)\} = 0 \end{aligned} \quad (103)$$

Hence $x^* = y^*$.

Conclusion and Recommendations

In this work, some new theorems on existence and uniqueness of fixed points of operators satisfying certain contractive conditions in a complete generalized metric space were established.

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