Generalization of Some Results in Linear Orthogonality Spaces

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Abstract
Let X and Y be linear orthogonality normed spaces and $T : X \to Y$ be orthogonally additive map. In this paper, we proved that T is continous if the graph of T $(G(T))$ is closed. Extension of some existing results to linear orthogonality spaces are also given.

Keywords: Sublinear Functional; Normed Space; Linearity; Additive Mappings.

Introduction

Let $(E,\perp)$ be linear orthogonality space, f an orthogonally additive function and $x, y \in E$. Rauf and Kanu (2015) proved that, under suitable assumptions on E, f is sublinear functional. Furthermore, if $T : X \to Y$ is orthogonality additive, where X and Y are linear orthogonality spaces, then T^{-1} is continous. Some useful results can be extended and generalised in linear orthogonality spaces via orthogonally additive mappings.

Definition (Linear Orthogonality Space)

Let E be a real linear space with $\dim E \geq 2$ and $\perp \subset E^2$ be a relation such that:

1. $x \perp y$ if and only if $y \perp x$ and $x \perp y$ for every $x \in E$ implies $y = 0$, or $y \perp x$ for all $y \in E$ implies $x = 0$;
2. if $x, y \in E \setminus \{0\}$ and $x \perp y$, then x and y are linearly independent;
3. if $x, y, z \in E$, $x \perp y$ and $x \perp z$ imply $x \perp y + z$;
4. if $x, y \in E$ and $x \perp y$, then $ax \perp by$ for every $a, b \in \mathrm{P}$; and
5. if P is a 2-dimensional subspace of E, $x \in P$ and $a \in \mathrm{P}^+$, then there is $y \in P$ with $x \perp y$ and $x + y \perp ax - y$. Then, $(E,\perp)$ is a linear orthogonality space.

Any linear space can be made into a linear orthogonality space if we define $x \perp 0$, $0 \perp x$ for all x, and for non-zero vectors x, y define $x \perp y$ iff x, y are linearly independent. A linear orthogonality space endowed with a norm is called a linear orthogonality normed space.

Lemma: If $f : E \to \mathrm{P}^+$ is orthogonally additive, then there is a $c \in \mathrm{P}^+$ with $f(x) = c\|x\|^2$.

Theorem 1: Let E be a linear orthogonality space. If $f : E \to \mathrm{P}$ is orthogonality additive and $f(x) \leq M\|x\|^2$ for all $x \in E$ and for some $M \geq 0$, then there is an $\alpha \in \mathrm{P}$ such that $f(x) = \alpha\|x\|$.

Theorem 2: Let E be a linear orthogonality space and let $f : E \to \mathrm{P}$ be orthogonally additive and satisfy $|f(x)| \leq M\|x\|$ for all $x\, inE$. Then f is a sublinear functional.

Theorem 3: Let $(X,\perp)$ and $(Y,\perp)$ be linear orthogonality normed spaces. Suppose that $T : X \to Y$ is orthogonally additive and $T \leq M\|x\|$ for all $x \in X$, $M \geq 0$. Then $T^{-1} : Y \to X$ is continuous.

Corollary: Let X and Y be linear orthogonality normed spaces. Suppose that $T : X \to Y$ satisfying $T(x) \leq M\|x\|$ for all $x \in E$ and $M \geq 0$ is:

1. orthogonally additive;

2. bijective;
3. continuous; and
4. linear.

Then, $T^{-1}: Y \to X$ is continuous.

The proof of the results in 2.2 to 2.6 can be sourced from Kanu and Rauf (2014).

Results

In this section, extension of some results to linear orthogonality spaces are established and proved.

Theorem 4: Let X and Y be linear orthogonality normed spaces and $T: X \to Y$ be orthogonally additive map. Then T is closed if and only if for arbitrary orthogonal sequences $\{x_n\} \in D(T)$, and $\{y_n\} \in R(T)$, with $x_n \to \sum a_n x_n$ and $Tx_n \to \sum a_n y_n$ we have:

(i) $\sum a_n x_n \in D(T)$, and

(ii) $T(\sum a_n x_n) = \sum a_n y_n$

Proof:

$(\Rightarrow)$ Suppose T is closed. Let $\{x_n\} \in D(T)$ and $\{y_n\} \in R(T)$ be such that $x_n \to \sum a_n x_n$ and $T(x_n) \to \sum a_n x_n$. We shall prove that

$$\sum a_n x_n \in D(T) \text{ and } T(\sum a_n x_n) = \sum a_n y_n.$$

Let $x, y \in X$. Then x and y can be expressed as

$$x = \sum a_n x_n \text{ and } y = \sum a_n y_n \in Y$$
$$x_n \to x \text{ and } Tx_n \to y$$

implies that

$$(x_n, Tx_n) \to (\sum a_n x_n, \sum a_n y_n)$$

Moreover, $(x_n, Tx_n) \in G(T)$ for each n and since $G(T)$ is closed, we have $(\sum a_n x_n, \sum a_n y_n) \in G(T)$. This implies that $\sum a_n x_n \in D(T)$ and $T(\sum a_n x_n) = \sum a_n y_n$ since T is orthogonally additive.

$(\Leftarrow)$ Suppose that whenever $x_n \in X$, $x_n \to \sum a_n x_n$ and

$T(\sum a_n x_n) \to \sum a_n y_n$ we have

$$\sum a_n x_n \in D(T) \text{ and } T(\sum a_n x_n) = \sum a_n y_n.$$

We want to prove that $G(T)$ is a closed subset of X x Y (i.e., T is a closed map).

Let $\{(x_n, Tx_n)\}$ be a sequence in $G(T)$ such that

$$(x_n, Tx_n) \to (\sum a_n x_n, \sum a_n y_n).$$

It suffices to prove that

$$x_n \to \sum a_n x_n \text{ and } Tx_n \to \sum a_n y_n.$$

By hypothesis

$$\sum a_n x_n \in D(T) \text{ and } Tx_n = \sum a_n y_n$$

and this implies that

$$(\sum a_n x_n, \sum a_n y_n) = (\sum a_n x_n, T(\sum a_n x_n)) \in G(T)$$

ans so, $G(T)$ is closed.

The folowing theorem is equivalent to the Close Graph Theorem.

Theorem 5: Let X and Y be linear orthogonality normed spaces and $T: X \to Y$ be orthogonally additive map and $T \leq M\|x\|$ for all $x \in X$, $M \geq 0$.. Suppose, the graph of T $(G(T))$ is closed. Then, T is continuous.

Proof:

Let $T: X \to Y$ be defined by $T(x) = \alpha\|x\|$.

By this definition, T is orthogonally additive. Let

$$F_1 : G(T) \to X \text{ and } F_2 : G(T) \to Y.$$

Then

$$G(T) = (x, \alpha\|x\|),$$

$$F_1(x, \alpha\|x\|) = x$$

And

$$F_2(x, \alpha\|x\|) = \alpha\|x\|.$$

Now, $G(T)$ is closed in X x Y implies $G(T)$ is a linear orthogonality space. Let

$$F_1(x, \alpha\|x\|) = 0 \text{ then } x = 0.$$

Therefore, F_1 is one-to-one and there exists F_1^{-1} on $R(F_1)$ and $F^{-1}: R(F_1) \to G(T)$ is linear. By Theorem 3, F^{-1} is continuous. Clearly, F_2 is also continuous. Hence,

$$T = F_2 o F_1^{-1} : X \to Y$$

is continuous since it is the composition of two continuous maps.

Theorem 6: Let X and Y be linear orthogonality normed spaces and let F be a family of bounded orthogonally additive operators from X to Y. Suppose for each $x \in X$ there exists a constant M_x such that $\|Tx\| \leq M_x$ forall $T \in F$. Then there exists a constant M such that $\|T\| \leq M$ forall $T \in F$.

Proof:

Let $W_n = \{x \in X : \|Tx\| \leq n \text{ forall } T \in F\}$ for each $n \in \mathrm{N}$. Each T is continuous since it is bounded and $W_n = \bigcap_{T \in F} \{x : \|Tx\| \leq n\}$ are closed sets since the set $\{x : \|Tx\| \leq n\}$ is closed.

By Baiye's Category theorem, we conclude that there exists $n_o \in \mathrm{N}$ and a ball $B_r(x_o) \subset X$ such that

$$\|T(x)\| \leq n_o \text{ forall } x \in B_r(x_o) \text{ and } T \in F.$$

Let $z, x_o \in X$. Then

$$\|T(z)\| = \|T((z + x_o) - x_o)\| = \|T(z + x_o) - T(x_o)\| \leq \|T(z + x_o)\| + \|T(x_o)\| \leq 2n_o.$$

This implies $-x_o \perp z + x_o$ since T is orthogonally additive.

That is, $\|T(z)\| \leq 2n_o$ on $B_r(o)$ and $\|z\| < r$ and $z + x_o \in B_r(x_o)$.

So, $\|Tx\| \leq \frac{2n_o}{r}$ forall $x \in B_1(o)$.

Thus, $\|T\| \leq \frac{2n_o}{r}$.

This implies that $\|T\| \leq M$ forall $T \in F$ where $M = \frac{2n_o}{r}$.

Theorem 6 is equivalent to the Uniform Boundedness Thoerem.

Theorem 7: Let X and Y be linear orthogonality normed spaces spaces and let $T_n \in B(X,Y)$ the family of bounded orthogonally additive operators from X to Y, for all $n \geq 1$. Suppose for each $x \in X$, $\{T_n(x)\}$ converges to a limit defined by Tx. Then,

(a) $sup\|T_n\| < \infty$

(b) $T \in B(X,Y)$

(c) $\|T\| \leq \lim inf\|T_n\|$

Proof:

Since $\{T_n(x)\}$ converges to Tx, $\{\|T_n(x)\|\}$ is bounded. Thus, there exists $M_x \geq 0$ such that

$$\|T_n(x)\| \leq M_x.$$

From Theorem 3.3, their exists, a constant M such that

$$\|T_n\| \leq M \text{ forall } n \in \mathrm{N}.$$

This implies that

$$sup_{n\geq 1}\|T_n\| < \infty,$$

and this establishes (a).

Let $x, y \in X$. Then by (L04), $ax \perp by$ for all $a, b \in R$ and

$$T(ax+by) = lim_{n\to\infty} T_n(ax+by) = lim_{n\to\infty}(aT_n(x) + bT_n(y)) = aT(x) + bT(y).$$

This implies that T is orthogonally additive and linear. For every $x \in X$ with $\|x\| = 1$, we have

$$\|T(x)\| = \lim_{n\to\infty}\|T_n(x)\| \leq \lim_{n\to\infty} sup\|T_n\|\|x\| = \lim_{n\to\infty} sup\|T_n\| \leq M.$$

Thus, $\|T\| \leq M$. So T is bounded and $T \in B(X,Y)$, establishing (b).

Finally,

$$\|T_n(x)\| \leq \|T_n\|\|x\|, n \geq 1$$

$$\lim_{n\to\infty} inf\|T_n(x)\| \leq (lim_{n\to\infty} inf\|T_n\|)\|x\|,$$

$$\|T(x)\| \leq (lim_{n\to\infty} inf\|T_n\|)\|x\|$$

for each $x \in X$. This implies that

$$\|T\| \leq \lim_{n\to\infty} inf\|T_n\|.$$

Conclusions

In this work, we generalized the following results in linear orthogonality spaces: closed grah theorem, Uniform boundedness theorem, Banach-Steinhaus theorem. Also, we characterized linear orthogonality spaces via orthogonally additive functions. For further references, see Baron and Ratz (1995), Birkhoff (1935), Brzdek (1997), James (1947), Kreyszig (1978), Saidi (2002) and Wlodzimiers and Justyna (2008). The results of this paper are remarkable and valuable in the concept of linear spaces.

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