

Series Solution of Euler-Bernoulli Beam Subjected to Concentrated Load

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Abstract

This paper investigates the series solution of Euler – Bernoulli beam with a concentrated load. The governing partial differential equation was transformed to Ordinary Differential Equation. Graphical representation of the various deflections of the beam with respect to varying parameters was depicted. It was shown that as the mass increases, the deflection also increases and as the mass of beam decreases the deflection also increases.

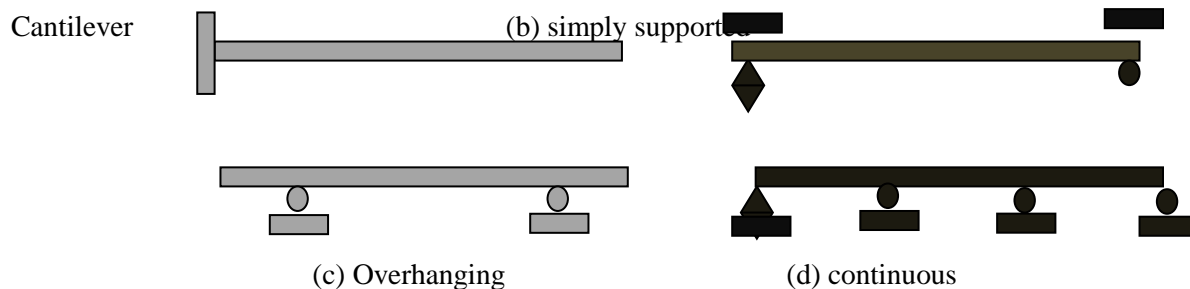
Keywords: Euler – Bernoulli beam; beam; Load; series solution; Concentrated Load.

Introduction

A beam is a structural element that is capable of withstanding load primarily by resisting bending. The bending force induced into the material of the beam as a result of the external loads, own weight, span and external reactions to these loads is called a bending moment (Abu-Hilal, 2003). Beams are characterized by their profile shape of cross-section, their length, and their material. Beams are traditionally descriptions of building or civil engineering structural element, but smaller structures such as truck or automobile frames, machine frames, and other mechanical or structural system contain beam structures that are designed and analyzed in a similar fashion (Kozien,2013).

Due to the increase of the speed of rail vehicles, the dynamic behavior of train/track interaction s getting more and more important. In the modeling of interaction between wheel and rail, the influence of contact force oscillation cannot be neglected. This behavior of the contact force can be caused by the periodic spacing of slippers or corrugation of wheel and rails. Our study will give an insight into the solution of Euler – Bernoulli beam subjected to concentrated load using a series solution (Inglis,1934; Lu, 2003; Usman, 2003; Mehri, Davar and Rahmani, Nguyen, 2011).

Historically beams were squared timbers but are also metal, stone, or combinations of wood and metal such as a flitch beam. Beams generally carry vertical gravitational forces but can also be used to carry horizontal loads e.g. loads due to an earthquake or wind or intension to resist rafter thrust as a tie beam or compression as a collar beam (Mehri, Davar and Rahmani, 2009). The loads carried but a beam is transferred to columns walls or girders, which then transfer the force to adjacent structural compression members. In light frame construction joists may rest on beams. In carpentry a beam is called a plate as in a sill plate or wall plate, beam as in a summer beam or dragon beam.



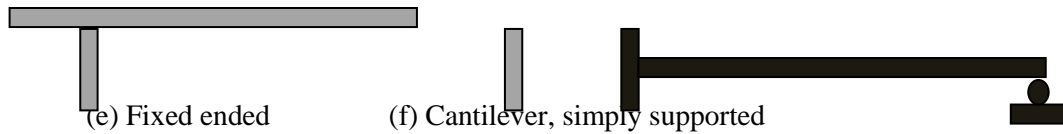


Figure 1.1: Type of Beams based on the manner in which they are supported.

Most beams in reinforce concrete buildings have rectangular cross sections, but a more efficient cross section for a beam is an I or it section which is typically seen in steel construction. Because of the parallel axis theorem and the fact that most of the material is away from the neutral axis the second moment or area of the beam increases, which in the beam increases the stiffness. An I – beam is only most efficient shape in one direction of bending: up and down looking at the profile as an I. If the beam is bent side to side, it functions as an II where it is less efficient. The most efficient shape for both direction in 2D is a box, however, the most efficient shape for bending in any direction is a cylindrical shell or tube. Efficiency means that for the same cross- sectional area (volume of beam per length) subjected to the same loading conditions the beam deflects less.

A thin walled beam is a very useful type of beam. The cross section of this walled beams is made up from this panels connected among themselves to create closed or open cross sections of a beam (structure). Typical closed sections include round square, and rectangular tubes. Open sections include I-beams. T-beams, L-beams, and so on. These walled beams exist because their bending stiffness per unit cross sectional area is much higher than that for solid cross sections such a rod or bar.

Mathematical Formulation

Consider the Vibration of a Euler – Bernoulli beam of length L. The Partial differential equation for the Vibration of the system is given as.

$$EI \frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} + K(x)w(x, t) = F(x, t) \quad (1)$$

Where

E = Young's modulus , I = Moment of inertia of the cross section

ρ = Density of the mass, A = Area of the cross section of the beam

EI = Rigidity of the beam, K = Winkler's Foundation

F(x,t) = Forced response

The forced response is given as;

$$F(x,t) = M \delta(x-vt) \quad (2)$$

where v = velocity of the load

t = time taken by the load

Substituting equation (3.2) into equation (3.1). the governing equation becomes

$$EI \frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} + K(x)w(x,t) = M\delta(x - vt) \quad (3)$$

The boundary conditions of Vibration of Simply supported beam takes the following form.

$$w(0,t) = 0 \quad w(L,t) = 0$$

$$\frac{\partial^2 w(0,t)}{\partial x^2} = 0 \quad \frac{\partial^2 w(L,t)}{\partial x^2} = 0 \quad (4)$$

with the initial conditions

$$w(x,0) = 0 \quad \text{and} \quad \frac{\partial w(x,0)}{\partial t} = 0 \quad (5)$$

Method of Solution

Assume a solution of the form

To solve the fourth order partial differential equation, we use finite fourier sine series:

$$w(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L} \quad (6)$$

and method of undetermined coefficient is used to solve time function $T(t)$.

$$f_n(t) = \int_0^L F(x,t) x_n(x) dx \quad (7)$$

with the inverse

$$F(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} f_n(t) X_n(x) \quad (8)$$

$$f_n(t) = \int_0^L M\delta(x - vt) \sin \frac{n\pi x}{L} dx \quad (9)$$

Going by the property of dirac delta function which says

$$\int_a^b \delta(x - v) f(x) = \begin{cases} f(v) & a < v < b \\ 0 & a < b < v \\ 0 & v < a < b \end{cases}$$

equation (3.9) becomes

$$f_n(t) = M \sin \left(\frac{n\pi v}{L} \right) t \quad (10)$$

$$F(x,t) = \frac{2M}{L} \sum_{n=1}^{\infty} \sin \left(\frac{n\pi v}{L} \right) t \sin \left(\frac{n\pi x}{L} \right) \quad (11)$$

$$EI \left[\sum_{n=1}^{\infty} T_n(t) \left(\frac{n\pi}{L} \right)^4 \sin \left(\frac{n\pi x}{L} \right) \right] + \rho A \sum_{n=1}^{\infty} \ddot{T}_n(t) \sin \left(\frac{n\pi x}{L} \right) + K \sum_{n=1}^{\infty} T_n(t) \sin \left(\frac{n\pi x}{L} \right) = f(x,t)$$

$$\left[EI \left(\frac{n\pi}{L} \right)^4 + K \right] \sum_{n=1}^{\infty} T_n(t) \sin \left(\frac{n\pi x}{L} \right) + \rho A \sum_{n=1}^{\infty} \ddot{T}_n(t) \sin \left(\frac{n\pi x}{L} \right) = 2 \frac{M}{L} \sin \left(\frac{n\pi v}{L} \right) t \sin \left(\frac{n\pi}{x} \right) K$$

divide through by $\rho A \sum_{n=1}^{\infty} \sin \left(\frac{n\pi x}{L} \right)$, we have

$$(13)$$

$$\left[EI \left(\frac{n\pi}{L} \right)^4 + K \right] \sum_{n=1}^{\infty} T_n(t) + \sum_{n=1}^{\infty} \ddot{T}_n(t) = \frac{2}{\rho A L} M \sum_{n=1}^{\infty} \sin \left(\frac{n\pi v}{L} \right) t$$

$$(14)$$

$$\text{let } N^2 = \frac{EI}{\rho A} \left(\frac{n\pi}{L} \right)^4 + k \text{ \& } C = \frac{2}{\rho A L} M$$

$$\ddot{T}(t) + N T(t) = C \sin \left(\frac{n\pi v}{L} \right) t$$

$$(15)$$

$$\text{let } w_n = \frac{n\pi v}{L}$$

$$\ddot{T}(t) + N^2 T(t) = C \sin w_n t$$

$$(16)$$

solvig the homogeneous part that is

$$\ddot{T}(t) + N^2 T_n(t) = 0$$

$$(17)$$

We have the solution in the following form

$$T_A(t) = A_n \sin Nt + B_n \cos Nt$$

$$(18)$$

Using the method of undetermined coefficient

$$\text{let } T_p(t) = D \sin w_n t + E \cos w_n t$$

$$(19)$$

Differentiating equation (19) and inserting back into equation (18) we have

$$-D w_n^2 \sin w_n t - w_n^2 E \cos w_n t + N^2 D \sin w_n t +$$

$$N^2 E \cos w_n t = C \sin w_n t$$

$$(20)$$

$$\text{Sin } w_n t : -w_n^2 D + N^2 D = C$$

$$(21)$$

$$\text{Cos } w_n t : -w_n^2 E + N^2 E = 0$$

$$(22)$$

From eqn. (B) $E = 0$

From eqn. (A) we have that

$$D(-w_n^2 + N^2) = C$$

$$D = \frac{C}{N^2 - w_n^2} \quad (23)$$

$$T_p = \frac{C}{N^2 - w_n^2} \sin w_n t \quad (24)$$

$$T_G = T_A + T_p \quad (25)$$

$$= A_n \sin Nt + B_n \cos Nt + \frac{C}{(N^2 - w_n^2)} \sin w_n t - C \quad (26)$$

Substituting into the initial condition

$$\sum_{n=1}^{\infty} T_n(0) \sin \frac{n\pi}{L} x = 0$$

and

$$\sum_{n=1}^{\infty} T_n'(0) \sin \frac{n\pi}{L} x = 0$$

$$T_n(t) = A_n \sin Nt + B_n \cos Nt + \frac{C}{(N^2 - w_n^2)} \sin w_n t$$

$$T_n'(t) = NA_n \cos Nt - NB_n \sin Nt + \frac{C}{(N^2 - w_n^2)} w_n \cos w_n t \quad (28)$$

$$T_n(0) = 0 = B_n$$

$$T_n'(0) = 0 = NA_n + \frac{C w_n}{(N^2 - w_n^2)}$$

$$A_n = -\frac{C w_n}{N(N^2 - w_n^2)} \quad (29)$$

Substituting the A_n & B_n into equation (26) we have

$$-\frac{C w_n}{(N^2 - w_n^2)} \sin Nt + \frac{C}{(N^2 - w_n^2)} \sin w_n t$$

$$T_n(t) = \frac{C}{(N^2 - w_n^2)} [\sin w_n t - \sin Nt] \quad (30)$$

$$W(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L}$$

$$W(x, t) = \frac{C}{(N^2 - w_n^2)} [\sin w_n t - \sin Nt] \sin \frac{n\pi x}{L}$$

Where

(31)

$$C = \frac{2 M}{\rho A L}$$

$$N = \sqrt{\frac{EI}{\rho A} \left(\frac{n\pi}{L}\right)^4 + k} \quad w_n = \sqrt{\frac{n\pi v}{L}}$$

Discussion

The beam was assumed to possess the following parameters: E (modulus of elasticity) = 2.10E10N, I (Moment of Inertia) = 8.33 x 10⁻¹⁷, V (velocity) = (0.1, 0.3, 0.5)m/s, K (Winkler foundation) = 0,5,10, L (length) = 1/192m⁴ Mass per unit length (ρA)=0.98kg/m and Mass of the load (M)=50; 70; 100kg

Figure 4.1 shows the deflection of beam at various values of the velocity, in this graph, the deflection w(x,t) is plotted against various values of v. It is noted that as velocity increases, the amplitude of the deflection increases. It is found that the response amplitude increases as the velocity increases. Figure 4.2 shows the deflection of beam at various values of K. It is found that the response amplitude is the same in spite of the increase in the value of K. Clearly, from the graph, the response amplitude increases with an increase in the value of K. Figure 4.3-4.5 shows the deflection of Beam at various values of mass. It is found that the response amplitude increases for each of the velocity of the load.

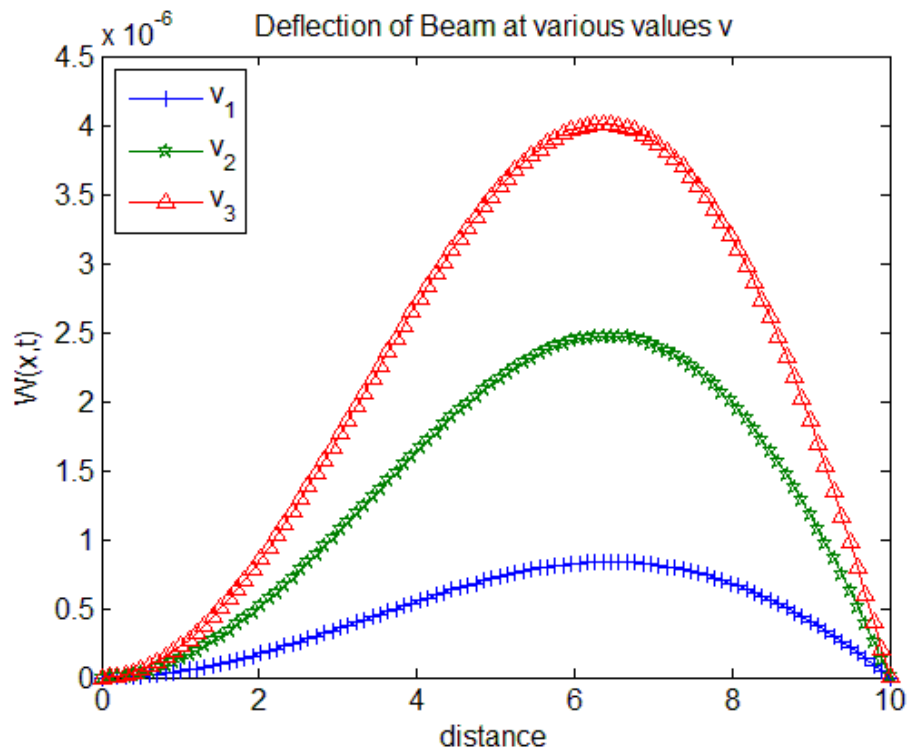


Figure 4.1: Deflection of Beam at various values of v

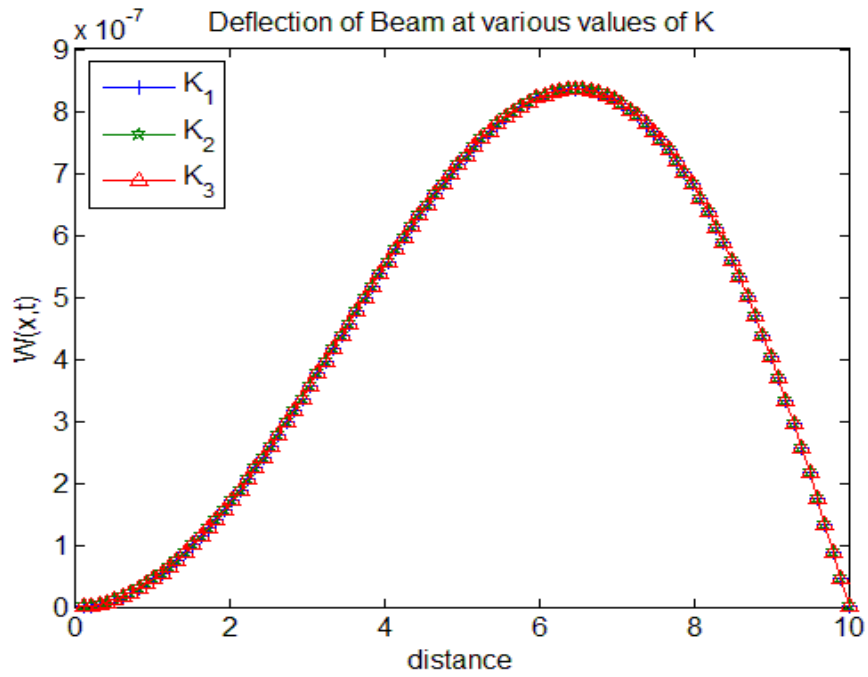


Figure 4.2: Deflection of Beam at various values of K

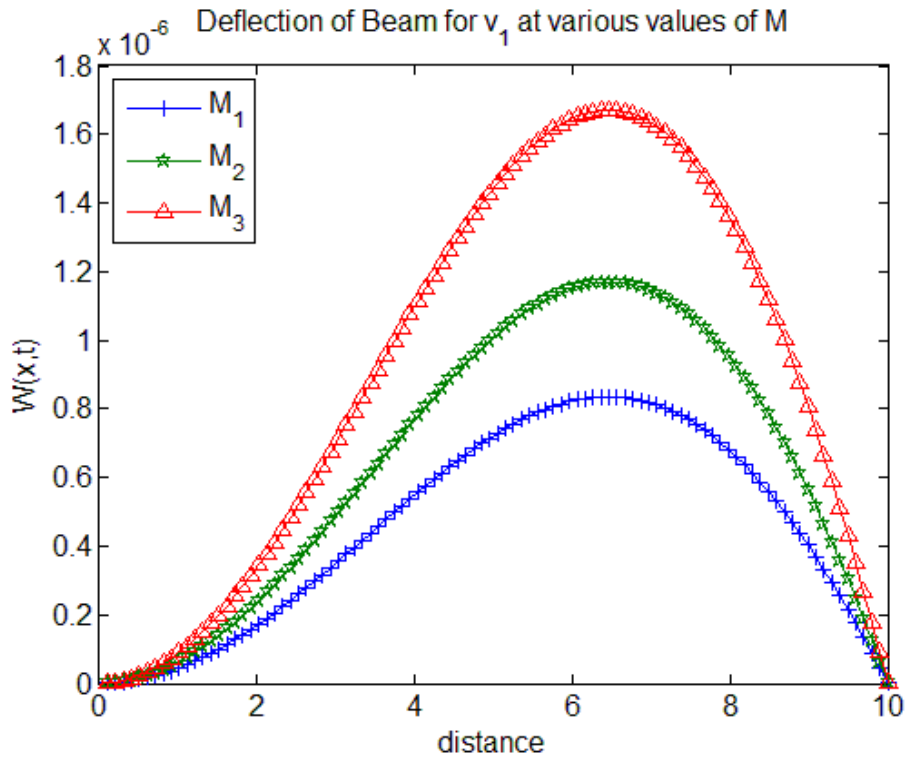


Figure 4.3: Deflection of Beam for $v=0.1$ at various values of Mass of the load

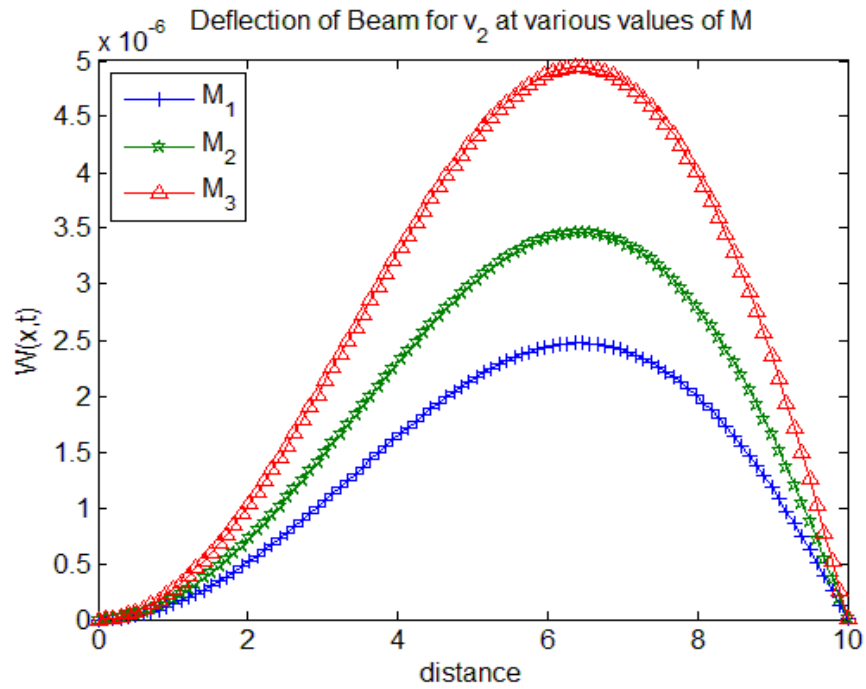


Figure 4.4: Deflection of Beam for $v=0.3$ at various values of mass of the load

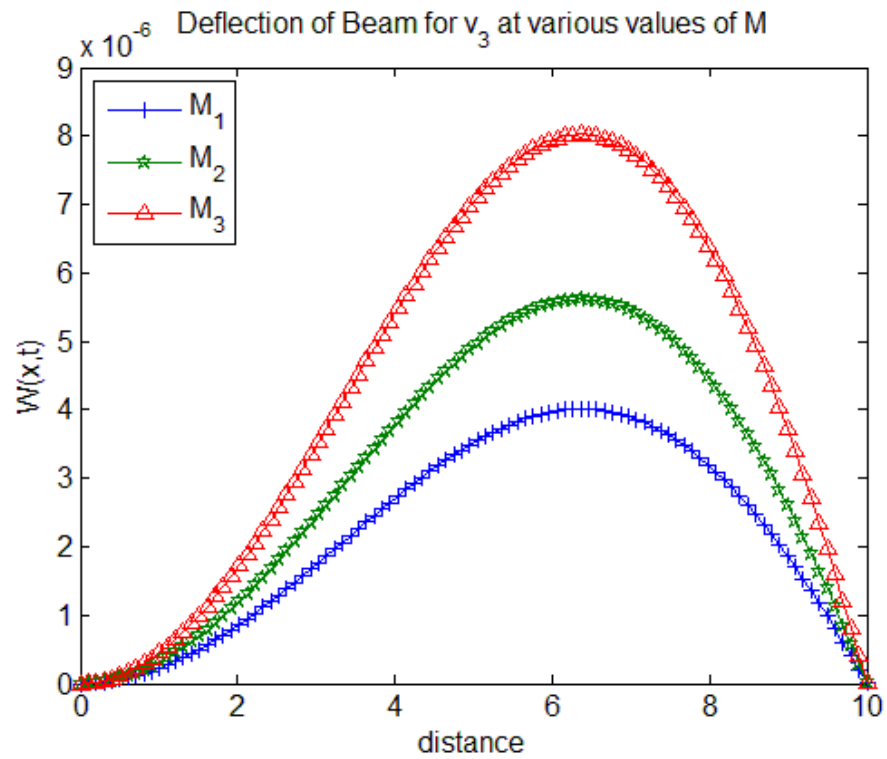


Figure 4.5: Deflection of Beam for $v=0.5$ at various values of mass of the load

Conclusions and Recommendations

The study considered the use of series solution to solve problem on Euler-Bernoulli beam subjected to concentrated load. It was also shown that the deflection of the beam increases as parameters of the system increases (Amplitude).

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