Mathematical Model of a Multilayer- Core Sandwich Beam

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Abstract
The subject of this research is to mathematically formulate the equation for the deflection of a seven layers sandwich beam. The sandwich beam consists of two facing, three cores and two binding layers between faces and core. The beam is subjected to a transverse load. The Mathematical model of the field of displacement of each layer of the beam which include a shear effect and bending moment is presented. The system of differential equation of equilibrium of the seven-layer sandwich beam is derived on the basis of the principle of stationary total potential energy. The modelled equations are analytically solved. The formula describing the deflection of the beam is obtained. The facing stress analysis of the beam is compared with analytically and theoretically. The comparison of the results obtained is in good agreement.

Keywords: Model; Beam; Layer; Shear Effect.

Introduction

The use of composite sandwich structure in aerospace and civil infrastructural applications has been increasing especially due to their extremely low weight that leads to reduction in the total weight and fuel consumption, high flexural and shear stiffness, and corrosion resistance (ASM Handbook, 1987). In addition, these materials are capable of absorbing large amount of energy under impact loads which result in high structural crash worthiness. In its simplest form a structural sandwich, which is a special form of laminated composite, is composed of two thin stiff face sheets and a thick light weight core bounded together. Sandwich structures are constructed in the form of beams, plates and panels. A sandwich structure will offer different mechanical properties with the use of different type of materials because the overall performance of sandwich structure depends on the constituents (Daniel, 2008). Hence optimum material choice is often obtained according to the design needs (Vinson, 1999). Various combination of cores and face sheets materials are utilized by researcher worldwide in other to achieved improved crash worthiness (Adams, 2006). The core material is normally low strength material, but its higher thickness provided the sandwich composite with high bending stiffness with overall low density. Open and closed cell structured foams like polyvinylchloride, polyurethethane, polyethylene or polystyrene foams and honey combs are commonly used as core materials. Open and closed metal foam can also be used as core materials. Laminate of glass or carbon fiber reinforced thermoplastic or mainly thermos set polymers are used as skin materials. Sheet metal is also used as skin materials in some cases. The core is bound with an adhesive or with metals components by brazing together.

Sandwich beams are composite system having low weight and high strength stiffness characteristics. Typical sandwich beam consists of two thin skin layers separated by a thick inner core. The use of thin, strong skin sheets adhere to thicker, light weight core materials has allowed industry to build strong, stiff, light and durable structures. When the skin and core are joined together, they function as a single structural component containing all the advantages of each component. Sandwich beams have high stiffness- to- weight and strength- to- weight ratio and are used as light weight load bearing components. Tensile and comprehensive stresses are mainly carried by the skins, while transverse shear stresses are predominantly experienced by the core. Materials such as steel and aluminium sheets are used for the skins. The main function of the core is to increase the flexural rigidity of the sandwich beam minimizing transverse deformation. Sandwich beams are generally thick structures in which the thickness is not negligible as compared to other dimension ns. Thus, shear deformation accounts for a significant amount of transverse deflection. The strength of a sandwich beam is determined by the resistance of the skin or the core to failure. Ideally, the skin should be design to resist axial stresses, whereas the core should be designed for limited shear. Selection of materials in designing the structural and/or mechanical components play an important role and is fixed based on strength, stiffness, cost and other mechanical properties such as hardness, toughness, wear resistance etc.

Jasion, Magnucka-Blandzi, Szye and Magnucki (2012),Jasion P and Magnucki K (2013) studied analytically, numerically and experimentally the global and local buckling and wrinkling of the face sheets of sandwich beams. They carried out a numerical analysis based on this solution in order to illustrate the effect of the spacing of intermediate supports and their compliance on the strain of continuous sandwich panels, and the joints fixing them to the steel framing. MagnuckaBlandzi and Magnucki (2007) optimized the sandwich beam with a metal foam core under strength and stability constrains. Magnucki *et al.* (2013) presented the strength analysis of a simply supported five-layer sandwich beams with a metal foam core. Smyczynski M.J and Magnucka-Blandzi (2015) analysed the stability of a five-layer sandwich beam with the use of broken line hypothesis of the deformation of a flat cross section of the beam. Grygorowicz M., Magnucki k. and Malinowski M. (2015) studied analytically and numerically the elastic buckling of a three-layered beam with variable mechanical properties of the core. Loja M.A, Barbosa J.I (2015) considered the use of different shear deformation theories to formulae different layer wise models, implemented through kringing based finite elements. They solved the dynamic problem in the frequency domain of soft-core sandwich beams. Dariushi, Sadighi and Shakeri (2016) presented an advanced high order sandwich panel theory for bending analysis of the moderately thick faced sandwich beams with a soft core. Smyczynski and Blandzi studied analytically, numerically and experimentally the strength of a three layers' sandwich beam with thin binding layer.

The aim of this work is subject to Mathematical modelling of the deflection of a multilayer core sandwich beam. Facing stress are calculated using the deflection equation for varying thickness of the core.

Mathematical Formulation of the Deflection of the Multilayer Core Sandwich Beam

The sandwich beam consists of seven layers: The upper and the lower faces, three cores and the binding layers between the faces and the core. The thickness of each of the face is t_f, thickness of each core is t_c, the central core is $t_{c1}$and the thickness of binding layer is t_b. The beam has length L, width b and depth H. The faces, cores and binding layers are isotropic. The Young's modulus E_f and poisson ratio ν_f are constant for each of the face also the Young's modulus E_c and poisson ratio ν_c are constant for each of the core. The beam carries a concentrated force F located at middle.

The analytical model of the cross section of the beam is formulated with regard to the broken line hypothesis. Grigolyk and Chulkov (1973) provided the first hypothesis of the cross-section deformation of sandwich structures, Carrera (2003) formulated the zigzag hypotheses for multilayer plates. The hypothesis for multi-layer structures is described in detailby Magnucka-Blandzi*et al.* (2016).

Displacements and strains

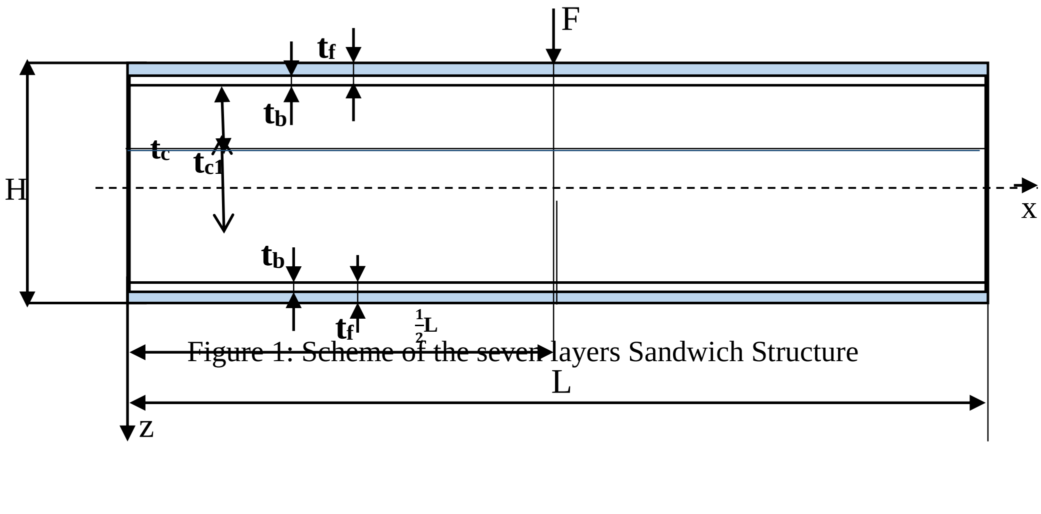

Figure 1: Scheme of the seven layers Sandwich Structure

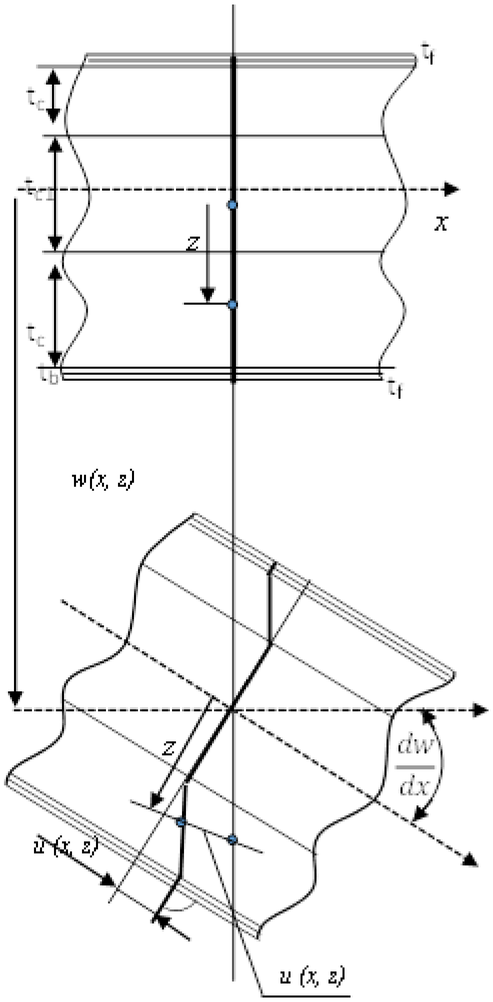

Figure 2: Scheme of deformation of the seven layers Sandwich Beam.

The field of displacements of the cross section of each layer of the beam is formulated with regard to the broken line hypothesis

1.The Upper facesheet $-(\frac{1}{2}+x_1+x_2+x_3) \le \xi \le -(\frac{1}{2}+x_2+x_3)$

$$u(x,\xi) = -t_{c_1}\left[\xi \frac{dw(x)}{dx} + \psi(x,\xi)\right] \quad (1)$$

2. The upper binding layer $-(\frac{1}{2}+x_2+x_3) \le \xi \le -(\frac{1}{2}+x_3)$

$$u(x,\xi) = -t_{c_1}\left[\xi \frac{dw(x)}{dx} + \psi(x,\xi)\right] \quad (2)$$

3. The upper core $-(\frac{1}{2}+x_3) \le \xi \le -\frac{1}{2}$

$$u(x,\xi) = -t_{c_1}\left[\xi \frac{dw(x)}{dx} + \psi(x,\xi)\right] \quad (3)$$

4. The middle core $-\frac{1}{2} \le \xi \le \frac{1}{2}$

$$u(x,\xi) = -t_{c_1}\xi\left[\frac{dw(x)}{dx} - 2\psi(x,\xi)\right] \tag{4}$$

5.The Lower Core $\frac{1}{2} \le \xi \le (\frac{1}{2} + x_3)$

$$u(x,\xi) = -t_{c_1}\left[\xi\frac{dw(x)}{dx} - \psi(x,\xi)\right] \tag{5}$$

6. The Lower binding layer $(\frac{1}{2} + x_3) \le \xi \le (\frac{1}{2} + x_2 + x_3)$

$$u(x,\xi) = -t_{c_1}\left[\xi\frac{dw(x)}{dx} - \psi(x,\xi)\right] \tag{6}$$

7. The lower facesheet $(\frac{1}{2} + x_2 + x_3) \le \xi \le (\frac{1}{2} + x_1 + x_2 + x_3)$

$$u(x,\xi) = -t_{c_1}\left[\xi\frac{dw(x)}{dx} - \psi(x,\xi)\right] \tag{7}$$

Where $x_1 = \frac{t_f}{t_{c_1}}$, $x_2 = \frac{t_b}{t_{c_1}}$, $x_3 = \frac{t_c}{t_{c_1}}$, are dimensionless parameters , $\xi = \frac{z}{t_c}$ dimensionless coordinate. $\psi = \frac{u(x)}{t_{c_1}}$ Dimensionless function of displacement which determine the field of displacement and the displacement $u(x)$.

Strain of the layers of the beam are defined by the geometric relation

1.The upper facesheet/ The lower facesheet

$$\varepsilon_x = -t_{c_1}\left[\xi\frac{d^2w(x)}{dx^2} \pm \frac{d\psi(x)}{dx}\right], \qquad \gamma_{xz} = 0 \tag{8}$$

2. The upper binding / The lower binding

$$\varepsilon_x = -t_{c_1}\left[\xi\frac{d^2w(x)}{dx^2} \pm \frac{d\psi(x)}{dx}\right], \qquad \gamma_{xz} = 0 \tag{9}$$

3. The upper core / The lower core

$$\varepsilon_x = -t_{c_1}\left[\xi\frac{d^2w(x)}{dx^2} \pm \frac{d\psi(x)}{dx}\right], \qquad \gamma_{xz} = 0 \tag{10}$$

4. The main core

$$\varepsilon_x = -t_{c_1}\xi\left[\frac{d^2w(x)}{dx^2} - 2\frac{d\psi(x)}{dx}\right], \qquad \gamma_{xz} = 2\psi(x) \tag{11}$$

The physical relationship for stress in each individual layers according to Hooke's law are

$\sigma_x = E\varepsilon_x$, $\tau_{xz} = G\gamma_{xz}$

1.The upper facesheet / The lower facesheet

$$\sigma_x = -t_{c_1}E_f\left[\xi\frac{d^2w(x)}{dx^2} \pm \frac{d\psi(x)}{dx}\right], \qquad \tau_{xz} = 0 \tag{12}$$

The bending moment in the cross section of the beam is

$$M_b(x)=\int_A \sigma_x z dA=-bt^3_{c_1}\left\{\left(2E_f c_{2f}+2E_b c_{2b}+2E_c c_{2c}+\frac{1}{12}E_{c_1}\right)\frac{d^2w}{dx^2}-\left(E_f c_{1f}+E_b c_{1b}+E_c c_{1c}+\frac{E_{c_1}}{6}\right)\frac{d\psi}{dx}\right\} \quad (13)$$

where

$$c_{2f}=\frac{1}{2}x_1^2+\frac{1}{3}x_1^3+\frac{1}{4}x_1+x_1x_2+x_1^2x_3+x_1x_2^2+x_1x_3^2+x_1x_3+x_1^2x_2+2x_1x_2x_3$$

$$c_{1f}=2x_1x_2+2x_1x_3+x_1+x_1^2$$

$$c_{2b}=\frac{1}{2}x_2^2+\frac{1}{4}x_2+x_2x_3^2+x_2x_3+\frac{1}{3}x_2^3+x_2^2x_3,\qquad c_{1b}=2x_2x_3+x_2+x_2^2$$

$$c_{2c}=\frac{1}{2}x_3^2+\frac{1}{3}x_3^3+\frac{1}{4}x_3\quad,\qquad c_{1c}=x_3^2+x_3,$$

And E_f, E_c, E_b, E_{c_1} are: Young's modulus of the faces E_f, Young's modulus of the cores E_c, Young's modulus of the bindings E_b, Young modulus of the middle core E_{c_1}.

Equations of equilibrium

The potential energy of the elastic strain of the beam is

$$U_\varepsilon=\frac{1}{2}\int(\varepsilon_x\sigma_x+\gamma_{xz}\tau_{xz})dv=\frac{1}{2}bt_{c_1}\int(U_{Ef}+U_{Eb}+U_{Ec}+E_{Ec_1})dx \quad (14)$$

Where

$$U_{Ef}=2E_f t^2{}_{c_1}\left[C_{2f}\left(\frac{d^2w}{dx^2}\right)^2-C_{1f}\left(\frac{d^2w}{dx^2}\right)\frac{d\psi}{dx}+x_1\left(\frac{d\psi}{dx}\right)^2\right]$$

$$U_{Eb}=2E_b t^2{}_{c_1}\left[C_{2b}\left(\frac{d^2w}{dx^2}\right)^2-C_{1b}\left(\frac{d^2w}{dx^2}\right)\frac{d\psi}{dx}+x_2\left(\frac{d\psi}{dx}\right)^2\right]$$

$$U_{Ec}=2E_c t^2{}_{c_1}\left[C_{2c}\left(\frac{d^2w}{dx^2}\right)^2-C_{1c}\left(\frac{d^2w}{dx^2}\right)\frac{d\psi}{dx}+x_3\left(\frac{d\psi}{dx}\right)^2\right]$$

$$U_{Ec_1}=\frac{1}{12}E_{c_1}t^2{}_{c_1}\left[\left(\frac{d^2w}{dx^2}\right)^2-4\left(\frac{d^2w}{dx^2}\right)\frac{d\psi}{dx}+4\left(\frac{d\psi}{dx}\right)^2\right]+4G_{c_1}\psi^2$$

The work of external load is

$W=\int_0^L qwdx,$ Where q is the intensity of the transverse load.

Equations of equilibrium

Two differential equations are derived based on the theorem of minimum potential energy $\delta(U_\varepsilon-W)=0$.

These equations are

$$a_{11}\frac{d^4w}{dx^4}-a_{12}\frac{d^3\psi}{dx^3}=\left(q-F_o\frac{d^2w}{dx^2}\right) \quad (15)$$

$$a_{12}\frac{d^3w}{dx^3}-a_{22}\frac{d^2\psi}{dx^2}+2G_{c_1}\frac{\psi}{t_{c_1}}=0 \qquad (16)$$

Equation (13) of the system is equivalent to equation (15). Therefore equation (15) can be written as

$$a_{11}\frac{d^4w}{dx^4}-a_{12}\frac{d^3\psi}{dx^3}=-\frac{M_b}{bt^3_{c_1}} \qquad (17)$$

Deflection of the Beam

The bending moment for the load case is written in the form

$M_b=\frac{1}{2}F(x)$.

By simple transformation of equations (15) and (16) we have :

$$(a_{12}{}^2-a_{11}a_{12})\frac{d^2w}{dx^2}+a_{11}\frac{2G}{t_{c1}}\psi-a_{12}\frac{F}{2bt^3_{c_1}}=0 \qquad (18)$$

Where

$$a_{11}=2E_f c_{2f}+2E_b c_{2b}+2E_c c_{2c}+\frac{1}{12}E_{c_1},a_{12}=E_f c_{1f}+E_b c_{1b}+E_c c_{1c}+\frac{1}{6}E_{c_1}$$

$$a_{12}=a_{21},a_{22}=2\left(E_f x_1+E_b x_2+E_c x_3+\frac{E_{c_1}}{6}\right)$$

The equation (18) is approximately solved by the Bubnuv-Galarkin method. The two unknown functions are assumed in the form of Fourier series.

$$\psi(x)=\psi_{11}\cos\frac{\pi x}{L}+\psi_{13}\cos\frac{3\pi x}{L}+.....+\psi_{1k}\cos\frac{k\pi x}{L}$$

$$w(x)=w_1\sin\frac{\pi x}{L}+w_3\sin\frac{3\pi x}{L}+.....+w_k\sin\frac{k\pi x}{L},k=1,3,5$$

As a result of orthogonalization process,

$w_k=\frac{2FL^3}{a_{11}\pi^4k^4}\left(\frac{1}{bt^3_{c_1}}\right)$. This gives the amplitude of deflection. (19)

The maximum deflection of the beam i.e for $L=\frac{x}{2}$ is

$$w\left(\frac{L}{2}\right)=\sum_{i=0}^{n}w_{2i+1} \quad , \text{ for } k=2i+1, i=0,1,2,........,n \qquad (20)$$

Discussion of Results

Calculations have been performed for a composite sandwich structure with the following component and dimensions under three-point bending and also the theoritical calculation using our modelled equation.

Two facesheets of composite glass fibres (T800/M300) polyster resin with nominal face thickness of 1mm. Honeycomb polyprophylene triple cores of total thickness of (20mm,22mm,24mm,26mm,28mm) and intermediate layers of composite glass fibres M450/Polysterwith nominal intermediate layer thickness of 0.05mm.L_1 = 120, L_2 = 300 , P = 931N , b = 35mm.G_c = 8MPa , E_c = 15MPa , E_f = 9162MPa, $V_f = V_b$ = 0.3 , E_b = 5500MPa , G_b = 2115MPa .

Analytical calculation σf: $\frac{P(L_2 - L_1)}{2t_f bd}$ (21)

Theoritical calculation σf : $zE_f\left(\frac{k\pi}{L}\right)^2 W_k$ (22)

Table 1

t(c)	20mm	22mm	24mm	26mm	28mm
Analytical calculation (σ_f)	113.7292162	103.8611714	95.56886228	88.50277264	82.40963855
Theoritical calculation (σ_f)	113.9678440	103.6260346	94.99347703	87.67937038	81.40347747
Calculated deflection (W_k)	22.07612065	18.40706462	15.58066453	13.35745754	11.40347747

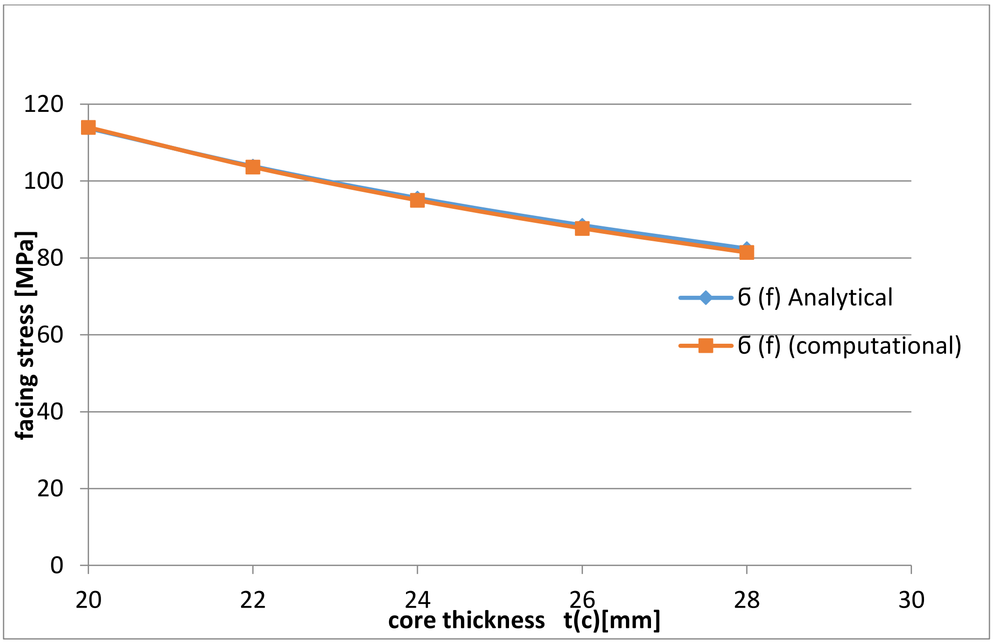

Fig 3: Facing stress against core thickness

Conclusions

The maximum deflection equation in the middle of the seven layers sandwich structure has been modelled in a way that the thickness and the material properties of any of the component on the deflection of the beam can be investigated. The facing stress is calculated analytically and using the theoretical modelled equation. The results obtained have been compared and it can be seen that they are in good agreement as the graphs overlapped. The finding also revealed from Table 1 that the deflection in the structure decreases with increase in the core thickness.

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