Seven Treatments from Mutually Orthogonal Latin Square of Order 7 in Symmetric Balanced Incomplete Block Designs

Saka A. J.[1], Adeleke M. O.[2], Adeleke B. L.[3] and Akeyede I.[4]

[1]Department of Mathematics, Obafemi Awolowo University, Nigeria

[2,3]Department of Statistics, University of Ilorin, Nigeria

[4]Department of Mathematics, Federal University Lafia, Nigeria ajsaka@oauife.edu.ng[1]; sakajamiu@gmail.com[1]

Abstract

A set of Mutually Orthogonal Latin Squares (MOLS) of order 7 gives rise to series of incomplete block designs, such as; balanced incomplete block designs (BIBD), and partially balanced incomplete block designs of two, three, four, five and six associate classes, that is PBIBD(k) with k =2,3,4,5 and 6. Two distinct Near-Resolvable BIBDs that are symmetric are equally obtained. All the aforementioned designs are constructed via orderly combinations of off-diagonal elements of a complete set of mutually orthogonal Latin squares (MOLS).

Keywords: Orthogonal Latin Square, symmetric, near-resolvable, incomplete block designs and partially balanced incomplete block designs

Introduction

Generally, Latin squares occur in many structures, including group multiplication tables, and Clayey tables. To be precise Latin squares are simply referred to as the multiplication tables of an algebraic structure called a quasi-group. Latin squares exhibit interesting combinatorial structures that have a wide range of areas of applications, such as design of experiment, combinatorics, tournament scheduling, finite geometries, and cryptography. One of the commonly known examples of Latin squares is Sudoku squares. Incidentally, Latin squares were first studied by a Swiss mathematician Leonhard Euler in 1782, while the problem of arrangement of 36 military officers in a clearly unique square was addressed (Anderson *et al.*, 2007). This question concerns a group of thirty-six officers of six different ranks, taken from six different regiments, and arranged in a square in a way such that in each row and column there are six officers, each of a different rank and regiment (Euler, (1782).

Eulers conjecture remained unsolved for over a hundred years before a step was taken towards confirming his predictions. In 1900 the French mathematician proved that there is no solution to the 36 officers problem by an exhaustive elimination of all possible cases, there by verifying Eulers conjecture for $n = 6$ (Tarry, 1900). This still left the orders $n = 10,14,18,\ldots$ unresolved, and it would only be 59 years later, when Indian mathematicians Raj Chandra constructed a Graeco-Latin square of order 22, that a special case of Eulers conjecture was disproven for the first time (Bose & Shrikhande, 1959). By this time the term Graeco-Latin square was no longer in use, the concept of orthogonality between Latin squares had taken its place. Also in 1959, the American mathematician Ernest Tilden Parker constructed two orthogonal Latin squares of order 10, and in 1960 Bose, Shrikhande and Parker finally disproved Eulers 177- year old conjecture for all other orders (Bose *et al.*, 1960; Parker, 1959).

This paper utilizes a set of Mutually Orthogonal Latin Squares (MOLS) of order 7 to construct a series of incomplete block designs, such as: balanced incomplete block designs (BIBD); partially balanced incomplete block designs of *m* associate classes (PBIBD (*m*)); and Near-Resolvable BIBDs [Abel and Furino (1996)].

Latin Squares

A Latin square of order n is an n by n array in which n distinct symbols from a symbol set n, say $\{1, 2, \ldots, n\}$ are arranged, such that each symbol occurs exactly once in each row and in each column.

Example 1

Latin squares of order 3 (L3), 4 (L4) and 5 (L5) are given below:

Given below are L1, L2, and L3 Latin squares of orders 3, 4 and 5 respectively.

L1=

1	2	3
2	3	1
3	1	2

L2=

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

L3=

5	1	2	3	4
2	3	4	5	1
4	5	1	2	3
1	2	3	4	5
3	4	5	1	2

Definition 1: (Reduced or Standard Latin Square)

A reduced or standard Latin square of order ***n*** is a Latin square that has its first row and column in the natural order (1,2,3,...,*n*).

Example 2

Reduced Latin square of order 4.

Given here as L2, is a reduced Latin square of order 4.

L2=

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

Definition 2: (Transpose of a Latin Square)

The transpose of a Latin square L of order n on the symbol set S, denoted by LT, is Latin square defined by LT (i,j)=L(j,i) for all $1\leq i,j \leq n$

Definition 3: (Symmetric Latin Square)

A Latin square **L** of order ***n*** is symmetric if L(i, j) = L^{T} (i, j) for all *i*, *j* *n* .

Example 3

Symmetric Latin square of order ***n***=5

Given here as L and *LT* are symmetric matrix and L its transpose *LT*

L=

5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3

and

5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3

Definition 4 (Idempotent Latin Square)

A Latin square L order ***n*** is idempotent if L(*i*,*j*) = *i* for *i*, *j* *n* or equivalently if the main diagonal of the Latin square is a transversal in natural order.

L =

1	7	6	5	4	3	2
3	2	1	7	6	5	4
5	4	3	2	1	7	6
7	6	5	4	3	2	1
2	1	7	6	5	4	3
4	3	2	1	7	6	5
6	5	4	3	2	1	7

Definition 5 (Mutually Orthogonal Latin squares)

Two Latin squares L1 and L2 of the same order, say *n* are mutually orthogonal if every ordered pair (*i*, *j*), *i*, *j* *n* appears exactly once when L1 and L2 are superimposed on each other.

Example 4

Mutually Orthogonal Latin Square of Order 4 Given here is a pair of orthogonal Latin square of order 4.

L1=

4	1	2	3
1	4	3	2
2	3	4	1
3	2	1	4

L2=

4	1	2	3
3	2	1	4
1	4	3	3
2	3	4	1

and S(L1,L2)=

(4,4)	(1,1)	(2,2)	(3,3)
(1,3)	(4,2)	(3,1)	(2,4)
(2,1)	(3,4)	(4,3)	(1,2)
(3,2)	(2,3)	(1,4)	(4,1)

Mutually Orthogonal Latin Squares of Order 7

Here, a complete set of mutually orthogonal Latin squares of order n = 7 which subsequently serves as a base for the construction of balanced incomplete block designs is presented. Meanwhile, a set of mutually orthogonal Latin squares is constituted by two more Latin squares of the same order, all of which are orthogonal to one another [Adeleke (1998)]. Furthermore, the sub (super) diagonal of a square is the set of elements directly below (above) the elements comprising the diagonal.

Example 5

A complete set of MOLS of order 7

A complete set of mutually orthogonal Latin squares of order 7 is presented below:

L1=

7	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5

L2=

7	1	2	3	4	5	6
2	3	4	5	6	7	1
4	5	6	7	1	2	3
6	7	1	2	3	4	5
1	2	3	4	5	6	7
3	4	5	6	7	1	2
5	6	7	1	2	2	4

L3=

7	1	2	3	4	5	6
3	4	5	6	7	1	2
6	7	1	2	3	4	5
2	3	4	5	6	7	1
5	6	7	1	2	3	4
1	2	3	4	5	6	7
4	5	6	7	1	2	3

L4=

7	1	2	3	4	5	6
4	5	6	7	1	2	3
1	2	3	4	5	6	7
5	6	7	1	2	3	4
2	3	4	5	6	7	1
6	7	1	2	3	4	5
3	4	5	6	7	1	2

L5=

7	1	2	3	4	5	6
5	6	7	1	2	3	4
3	4	5	6	7	1	2
1	2	3	4	5	6	7
6	7	1	2	3	4	5
4	5	6	7	1	2	3
2	3	4	5	6	7	1

L6=

7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1
1	2	3	4	5	6	7

Note that, since Latin squares L1, L*2*,..,L6 are pairwise orthogonal, they are therefore said to constitute a complete set of MOLS of order 7.

Definition 6: (A Complete Set of MOLS)

A set $t \geq 2$ MOLS of order n is called a complete set if $t = n$-1 or a set of n-1 MOLS of order n is called a complete set of MOLS.

Theorem 1: $N(n) \leq n-1$ for each $n \geq 2$, where n is the order of the Latin square while $n-1$, is the complete set of MOLS.

Theorem 2: Let Li(i=1,2,…,6) be a set of orthogonal Latin squares of order 7, if an initial block of size 3 is formed with the upper diagonal of length 2 or 3 of *L1* and *L2* and complementary block (mod 7) are generated with this initial block then the design obtained is a BIBD.

Theorem 3: Let L*i*(i=1,2,. ,6) be a set of orthogonal Latin squares of order 7 and suppose the main S+k,

$1 \leq k \leq 5$ denotes the set of elements on the main diagonals with distance $\pm k$ from the main diagonal of *Li*. Define the sets *Du ,i* = (*Sk ,i*), *Dl ,i* = (*S-k, i*), a collection of the sets $S \pm k$:

Case 1 (Latin Square on itself) for a fixed i, define (*Du ,i , Dl ,i*) := *Du ,i* *Dl ,i* ,$1 \leq i \leq 7$.

Case 2 (General) for each *i*, ($1 \leq i \leq 7$), and ($1 \leq i \leq 7$), *j* *i*

(*Du ,i , Dl , j*) :=

[*Du ,i* *Du* ,j, *Du ,i* *Dl* ,j, *D*l,*i* *Du* ,j, *D*l,*i* *D*l,j], the pattern formed by the combinations (*Du ,i ,Dl , j*) inboth cases is of the form BIBD provided both conditions below are satisfied:

$rv = kb$ (1)

$\lambda(v-1) = r(k-1)$ (2)

Using Theorem 3 the following initial blocks were obtained:

L2u = (5,1) and *L2l* = (3,6) then (L2, L2) = (5,1,3,6)

L2u = (4,7,3) and *L2l* = (3,6) then (L2, L2) = (4,7,3,6)

L1u = (5,7) and *L2u* = (5,1) then (L1, *L*2) = (5,7,1)

L2u = (4,7,3) and *L3l* = (5,2,6) then (*L*2, *L*3) = (4,7,3,5,2,6)

Construction of Balanced Incomplete Block Designs (BIBD)

In this section, a full specification of four of the several incomplete block designs constructed is effected by the cyclic development of the appropriate initial blocks. A summary of several other designs that were constructed are hereby presented:

Constructed Designs

Design 1: PBIBD(3) with the Initial Block (5,1,3,6)

A partially balanced incomplete block design with three associate class, and parameters $v = 7$, $r = 4$, $b = 7$, $k = 4$, $\lambda 1 = 1$, $\lambda 2 = 2$, $\lambda 3 = 3$, is as given as follows:

(5, 1, 3, 6)

(6, 2, 4, 7)

(7, 3, 5, 1)

(1, 4, 6, 2)

(2, 5, 7, 3)

(3, 6, 1, 4)

(4, 7, 2, 5)

Design 2: PBIBD(2) with the Initial Block (4,7,3)

A partially balanced incomplete block design with two associate class, with parameters $\boldsymbol{v} = 7$, $r = 3$, $b = \mathbf{7}$, $k = \mathbf{3}$, $\lambda 1 = 1$, $\lambda 2 = 2$, is given as follows:

(4, 7, 3)

(5, 1, 4)

(6, 2, 5)

(7, 3, 6)

(1, 4, 7)

(2, 5, 1)

(3, 6, 2)

Design 3: BIBD with the Initial Block (5, 7, 1)

A balanced incomplete block design with parameters $\boldsymbol{v} = 7$, $r = 3$, $b = \mathbf{7}$, $k = \mathbf{3}$, $\lambda = 1$is given as follows:

(5, 7, 1)

(6, 1, 2)

(7, 2, 3)

(1, 3, 4)

(2, 4, 5)

(3, 5, 6)

(4, 6, 7)

Design 4: Near Resolvable BIBD with the Initial Block (4, 7, 3, 5, 2, 6)

A near resolvable balanced incomplete block design with parameters $\boldsymbol{v} = 7$, $r = 6$, $b = \mathbf{7}$, $k = 6$, $\lambda = 5$ is given as follows:

(4, 7, 3, 5, 2, 6)

(5, 1, 4, 6, 3, 7)

(6, 2, 5, 7, 4, 1)

(7, 3, 6, 1, 5, 2)

(1, 4, 7, 2, 6, 3)

(2, 5, 1, 3, 7, 4)

(3, 6, 2, 4, 1, 5)

Design Evaluation

Table 1: Summary of the Designs Constructed on BIBDs, v = 7, 3 $\leq r \leq 6$

S/N	*V*	*B*	*r*	*K*	λ	λ1	λ 2	λ 3	Efficiency Factor	Initial block
1	7	7	3	3	–	1	2	–		(4, 7, 3)
2	7	7	3	3	–	1	2	–		(4, 2, 7)
3	7	7	3	3	–	1	2	–		(2, 7, 5)
4	7	7	4	4	–	1	2	3		(3, 6, 2, 5)
5	7	7	4	4	–	1	2	3		(5, 1, 3, 6)
6	7	7	3	3	1	–	–	–	0.78	(5, 7, 1)
7	7	7	3	3	1	–	–	–	0.78	(5, 7, 4)
8	7	7	3	3	1	–	–	–	0.78	(3, 6, 4)
9	7	7	4	4	2	–	–	–	0.88	(5, 7, 4, 3)
10	7	7	4	4	2	–	–	–	0.88	(4, 6, 1, 7)
11	7	7	6	6	5	–	–	–	0.97	(4, 6, 1, 2, 7, 5)
12	7	7	6	6	5	–	–	–	0.97	(4, 7, 3, 5, 2, 6)

Conclusion

The set of Mutually Orthogonal Latin Squares (MOLS) of order 7 was utilized to obtain series of initial blocks which were made to undergo the operation of cyclic development, such that the resulting values satisfied the arithmetic operation in modulo 7. This method indeed gave rise to a new method of construction of incomplete block designs, such as balanced incomplete block designs, and partially balanced incomplete block designs, with *m* associate classes, for *m* = 1,2,3 (BIBD, PBIBD (2) and PBIBD (3)) and with *k*= 3,4,5 & 6. Also, two distinct Near-Resolvable BIBDs that are symmetric were also constructed. All the designs constructed are with fewer numbers of blocks when compared.

References

Abel, R. J. R., Furino., S. C. (1996). The CRC Handbook of Combinatorial Design, Resolvable and near resolvable designs. CRC Press, Florida.

Adeleke, B. L. (1998). Resolvable Row-Column Designs: Constructions. An unpublished PhD thesis, Department of Statistics, University of Ilorin, Nigeria.

Anderson, I., Colbourn, C. J., Dinitz, J. H., & Griggs, T. S. (2007). The handbook of combinatorial designs (2nd edn), CRC Press, Florida.

Bose, R. C., Shrikhande, S. S. (1959). On the falsity of Eulers conjecture about the non- existence of two orthogonal Latin squares of order 4t + 2, Proc Natl Acad Sci USA, 45: 734-7.

Bose, R. C., Shrikhande, S. S., & Parker, E. T. (1960). Further results on the construction of mutually orthogonal Latin squares and the falsity of Eulers conjecture. Can J Math 7: 189-203.

Euler, L. (1782). Searching for a new kind of magic quarry (Recherches sur une nouvelle espece de quarresmagiques). Zeeland Soc Sci Flushing 9.

Parker, E. T. (1959). Orthogonal Latin squares, Proc Natl Acad Sci 45: 859-62.

Tarry, G. (1900). The problem of 36 o Ciers (Le probleme des 36 o ciers, ComptesRendus Association Franqaise Pour Advancement des Science). French Assoc Revised Acc Adv Sci 29: 170203.