Solving Linear and Nonlinear Integro-Differential Equations Using Modified Adomian Decomposition Method

Morufu O. Olayiwola[1], Kamilu A. Adedokun[2], Alagbe W. Gbolagade[3]
[1,2,3]Department of Mathematical Sciences, Faculty of Basic and Applied Sciences, Osun State University, Nigeria
[1]olayiwola.oyedunsi@uniosun.edu.ng

Abstract
In this paper, we used the new modification of Adomian Decomposition Method (MADM) to obtain the approximate solutions for the linear and nonlinear integro-differential equations. Illustrative examples have been discussed to demonstrate the validity, reliability of this new modification method. The rate at which the numerical solution converges to the exact solution is very fast. The method was implemented on Maple 18 platform. The method is elegant and therefore recommended for solving strongly nonlinear integro-differential equations that arise in physical sciences and engineering.

Keywords: Adomian decomposition method; Fredholm Integro-differential equations; Volterra Integro-differential equations.

Introduction

Kamel and Fathi (2005) reported that many problems in mathematical physics, theory of elasticity, visco dynamics, fluid and mixed problems of mechanics of continuous media can be reduced to the integral equation. Analytical methods for the solution of nonlinear integro-differential equations are usually hard, Anjan (2000, 2003), if not impossible and consequently, exact solutions are rather difficult to find. Several numerical methods have been used for the solution of such types of equations such as the Finite differences Zhaoand & Corless (2006), Tau method Abbasbandy & Taati (2003), Haar wavelet series technique Sekar & Jaisankar (2013), He's homotopy perturbation method Biazar et al. (2009), Biazar & Eslami (2011), Eslami (2014), Biazar et al. (2009), Eslami & Mirzazadehi (2014), Legendre polynomials and Block-pulse functions approach as contained in Maleknejad et al. (2011), the Adomian decomposition method, and the new modification of Adomian decomposition method Wazwaz (2011). For the solution of integral equations, Adomian (1988,1991) presented the so-called Adomian decomposition method (ADM). Wazwaz (2001) extended the method to include the solution of Volterra integral equation and the boundary value problems for higher order integro-differential equations.

In recent years, Olayiwola et al. (2018), Rabbani & Zarali (2012), Hendi & Bakodah (2012), Manafianheris (2012) and Alao et al. (2014) did some work on the solutions of Volterra-Fredholm integro-differential equations.

This method reduces the size of computation, while increasing the accuracy of the solution. It also separates the equation to be solved into two portions: linear and nonlinear. The solution generated by this method is in a series form whose terms are determined by a recursive relation using Adomian polynomials.

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda_i y_i\right)\right]_{\lambda=0}, n \geq 0 \quad (1)$$

where A_n denotes the Adomian polynomials of degree n and $\sum_{i=0}^{\infty} y_i(x,t)$ is the solution of the problem, $N(u)$ is the nonlinear term in the equation.

Recently, a new modification of Adomian decomposition method (NMADM) for finding exact solution of linear integral equations is presented by Hossein *et al.* (2014). A new reliable modification of the ADM is proposed and applied for the solution of the Volterra and Fredholm integral equations in Bakodah et al. (2017).

In this paper, the NMADM is applied for the solution of linear and nonlinear integro-differential equations.

Some examples are given to illustrate the validity, reliability of this new modified method. The results reveal that the presented modified method is very simple and effective.

Adomian Decomposition Method for Solving Nonlinear Integro-Differential Equations

The nonlinear fredholm integro-differntial equations of the second kind are given by:

$$y''(x) = f(x) + \int_a^b K(x,t)[R(y(t) + N(y(t)))]dt,$$

$$y(x_0) = \alpha_1, y'(x_0) = \alpha_2 \tag{2}$$

and the nonlinear fredholm integro-differential equations of the second kind are given by:

$$y''(x) = f(x) + \int_a^b K(x,t)[R(y(t) + N(y(t)))]dt$$

$$y(x_0) = \alpha_1 \quad \text{and} \quad y'(x_0) = \alpha_2 \tag{3}$$

where $y''(x)$ is the second derivative of the unknown function $y(x)$ that will be determined, $K(x,t)$ is the Kernel of the integral equation, $f(x)$ is an analytic function, $R(y(t))$ and $N(y(t))$ are linear and nonlinear function of y, respectively. Equation (2) and (3) can be written in an operator form:

$$L(y(x)) = f(x) + \int_a^b K(x,t)[R(y(t)) + N(y(t))]dt \tag{4}$$

$$L(y(x)) = f(x) + \int_a^x K(x,t)[R(y(t)) + N(y(t))]dt \tag{5}$$

The inverse operator L^{-1} is therefore considered an n-fold integral operator defined by $L^{-1}(.) = \int_0^x (.)dx$, operating with $L^{-1}x$ to both sides of (4) and (5) and using the initial condition, we have :

$$y(x) = g(x) + L_x^{-1}(y(x)) = f(x) + \left(\int_a^b K(x,t)[R(y(t)) + N(y(t))]dt\right) \tag{6}$$

and

$$y(x) = g(x) + L_x^{-1}(y(x)) = f(x) + \left(\int_a^x K(x,t)[R(y(t)) + N(y(t))]dt\right) \tag{7}$$

where $g(x)$ included the $L_x^{-1} f(x)$ and the initial conditions.

The standard Adomian decomposition method defines the solution $y(x)$ by the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{8}$$

Where $y_n(x)$ has to be determined sequentially upon the following algorithm:

$$y_0(x) = g(x) \tag{9}$$

$$y_{m+1}(x) = L_x^{-1}\left(\int_a^b K(x,t)[R(y_m(t)) + A_m(y(t))]dt\right), m \geq 0 \tag{10}$$

Where A_m, $m \geq 0$ are the Adomian polynomials.

A New Adomian Decomposition Method for Solving Nonlinear Integro-Differential Equations

A modified Adomian decomposition method (NMADM) provides the exact solution by using a single iteration. Only y_0 and y_1, with one exact solution(s) after one iteration are discussed. The solution is usually a unique solution, but in the present work, an example that gives two solutions is presented. In this method, the rate of convergence is accelerated. The ADM usually gives one solution among other solutions.

However, as will be seen in an example presented, the method can give more than one solutions. To achieve the goal equations (6) and (7) are rewritten as:

$$y(x)=\sum_{m=0}^{N}a_m v_m(x)-\sum_{m=0}^{N}a_m v_m(x)+g(x)+L_x^{-1}\left(\int_a^b K(x,t)[R(y(t))+N(y(t))]\right)dt \quad (11)$$

$$y(x)=\sum_{m=0}^{N}a_m v_m(x)-\sum_{m=0}^{N}a_m v_m(x)+g(x)+L_x^{-1}\left(\int_a^x K(x,t)[R(y(t))+N(y(t))]\right)dt \quad (12)$$

where $a_m, m=0,1,2,...,N$ are called the accelerating components of the parameter, $a_m v_m, m=0,1,2,...,N$ are selective functions.

Furthermore, the number of the terms in y_0, namely N is small in many practical problems.
Recall that the modified decomposition method is established based on the assumption that the function

$$f(x)=\sum_{m=0}^{N}a_m v_m(x)-\sum_{m=0}^{N}a_m v_m(x)+g(x) \quad (13)$$

can be divided into two parts, namely $f_1(x)$ and $f_2(x)$

where $f_1(x)=\sum_{m=0}^{N}a_m v_m(x)$ and $f_2(x)=-\sum_{m=0}^{N}a_m v_m(x)+g(x)$

Accordingly, a slight variation was proposed only on the components y_0 and y_1. It is suggested that only the part $f_1(x)$ be assigned to the zeroth component y_0, whereas the remaining part $f_2(x)$ be combined with the other terms given in (11) and (12) to define y_1. Consequently the modified recursive relation.

$$y_0(x)=\sum_{m=0}^{N}a_m v_m(x) \quad (14)$$

$$y_1(x)=-\sum_{m=0}^{N}a_m v_m(x)+g(x)+L_x^{-1}\left(\int_a^b K(x,t)[R(y_0(t)+A_0(y(t)))]dt,\right) \quad (15)$$

$g(x)=L_x^{-1}(f(x))$

and

$$y_m(x)=L_x^{-1}\left(\int_a^b K(x,t)[R(y_{m-1}(t))+A_{m-1}(y(t))]dt\right), m\geq 2 \quad (16)$$

was developed.

Numerical Applications

In this section, several examples are solved to illustrate this method for linear and nonlinear integro-differential equations

Linear Integro-Differential Equations (LIDE)

Example 1: consider the linear fredholm integro-differential equation:

$$y'(x) = 2xe^x + x^2e^x + 2x - xe + \int_0^1 xy(t)dt, \quad y(0) = 0 \tag{17}$$

With the exact solution $y(x) = x^2e^x$

$y_0(x)$ is chosen as:

$$y_0(x) = a_0e^x + a_1xe^x + a_2x^2e^x = \sum_{m=0}^{2} a_m x^m e^x \quad , \quad f(x) = 2xe^x + x^2e^x + 2x - xe$$

in view of equation (15) we have

$$y_1(x) = -y_0(x) + g(x) + \int_0^x \left(\int_0^1 (x) y_0(t) dt \right) dx = 0$$

Now, $a_m, m = 0,1,2$ are found such that $y_1(x) = 0$, if $y_1(x) = 0$ then the exact solution will be obtained as $y(x) = y_0(x)$.

Hence for all values of x we have

$$-a_0e^x - a_1xe^x - a_2x^2e^x + x^2e^x + x^2 - \frac{1}{2}x^2e + ea_1x - 2a_2xe + a_0x - 2a_1x + 6a_2x = 0$$

Collecting terms involving power of x

$$x^0e^x : -a_0 = 0, \quad a_0 = 0$$

$$x^1e^x : -a_1 = 0, \quad a_1 = 0$$

$$x^2e^x : -a_2 + 1, \quad a_2 = 1$$

We then have $y_1(x) = x^2e^x$ is the solution, which is the same as the exact solution.

Example 2: consider the linear Volterra integro-differential equation:

$$y''(x) = \sin(x) - \cos(x) + \frac{1}{6}x^3 - \int_0^x (x-t)y(t)dt, \quad y(0) = y'(0) = 1 \tag{18}$$

with the exact solution $y(x) = x + \cos x$

$y_0(x)$ is chosen as :

$$y_0(x) = a_0 + a_1x + a_2x^2 = \sum_{m=0}^{2} a_m x^m \quad , \quad f(x) = (x - \frac{x^3}{3!} + \frac{x^5}{5!}) - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^3}{3!}$$

In view of equation (15) we have,

$$y_1(x) = -a_2x^2 - xa_1 - a_0 + \frac{1}{6}x^3 + \frac{1}{5040}x^7 - \frac{1}{362880}x^9 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8$$

$$-\frac{1}{3024}a_5x^4 - \frac{1}{1080}a_4x^9 - \frac{1}{840}a_3x^7 - \frac{1}{360}a_2x^6 + \frac{1}{120}a_1x^5 - \frac{1}{24}a_0x^4 + 1 + x$$

Now, $a_m, m = 0,1$ and 2 are found such that $y_1(x) = 0$. If $y_1(x) = 0$ then $y_2(x) = y_3(x) = y_4(x) = ... = 0,$ and the exact solution will be obtained as $y(x) = y_0(x)$. Hence for all values of x we have ,

$$-a_2x^2 - xa_1 - a_0 + \frac{1}{6}x^3 + \frac{1}{5040}x^7 - \frac{1}{362880}x^9 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8$$

$$-\frac{1}{3024}a_5x^4 - \frac{1}{1080}a_4x^9 - \frac{1}{840}a_3x^7 - \frac{1}{360}a_2x^6 + \frac{1}{120}a_1x^5 - \frac{1}{24}a_0x^4 + 1 + x = 0$$

Collecting terms involving power of x :

$$x^0 : -a_0 + 1 = 0, \qquad a_0 = 1$$

$$x^1 : -a_1 + 1 = 0, \qquad a_1 = 1$$

$$x^2 : -a_2 - \frac{1}{2} = 0, \qquad a_2 = -\frac{1}{2}$$

We need to neglect the remaining terms like $x^3, x^4, ..., x^9$

$$y_1(x) = a_0 + a_1x + a_2x^2$$

$$y_1(x) = x + 1 - \frac{x^2}{2!}$$

So we need to increase N and let

$$y_0(x) = \sum_{m=0}^{\infty} a_m x^m$$

$$y_1(x) = -\sum_{m=0}^{\infty} a_m x^m + 1 + x + \frac{x^2}{2!} + \int_0^x \int_0^x \left(\int_0^x (x-t) \sum_{m=0}^{\infty} a_m t^m dt \right) dx dx$$

Therefore, the numerical solution is easily obtained as $y(x) = x + \cos x$ which satisfies both the conditions, and the equation.

Nonlinear Integro-Differential Equations (NIDE)

Example 3: consider the nonlinear fredholm integro-differential equation:

$$y^1(x) = 2x - \frac{1}{5}x^2 + \frac{1}{6} + \int_0^1 (x^2 - t) y^2(t) dt, \quad y(0) = 0 \tag{19}$$

with the exact solution:

$$y(x) = x^2$$

$y_0(x)$ is chosen to be :

$$y_0(x) = a_0 + a_1x + a_2x^2 = \sum_{m=0}^{2} a_m x^m, \; f(x) = 2x - \frac{1}{5}x^2 + \frac{1}{6}$$

In view of equation (5) we have:

$$y_1(x) = -y_0(x) + g(x) + \int_0^x \left(\int_0^1 (s^2 - t) A_0(t) dt \right) ds = 0, \; f(x) = 2x - \frac{1}{5}x^2 + \frac{1}{6}$$

where A_m are the well–known Adomian polynomials. The first term of Adomian polynomials that represents the linear operator $y^2(x)$ is $A_0(x)=y_0^2(x)$. Now we find $a_m, m=0,1,2.$

Hence for all values of x we have,

$$y_1(x)=-a_2x^2-a_1x+a_0+x^2-\frac{x^3}{15}+\frac{x}{6}+\frac{1}{3}\left(\frac{2}{3}a_0a_2+\frac{1}{3}a_1^2+\frac{1}{2}a_1a_2+a_0a_1+\frac{1}{5}a_2^2+a_0^2\right)x^3$$

$$+\frac{1}{6}a_2^2x-\frac{1}{4}a_1^2x-\frac{2}{5}a_1a_2x-\frac{2}{3}a_0a_1x-\frac{1}{2}a_0a_2x-\frac{1}{2}a_0^2x=0$$

which means that:

$$a_0=0, a_2=1, -5a_1^2-8a_1-20a_1=0, \qquad a_1=0 \text{ or } a_1=-\frac{28}{5}.$$

So, the solution which is the same as the exact solution and satisfies both the equation and the conditions will be $y(x)=x^2$, while the solution $y(x)=-\frac{28x}{5}+x^2$ does not.

Example 4: Consider the linear volterra integro-differntial equations.

$$y'(x)=e^x+\frac{1}{2}-\frac{1}{2}e^{2x}+\int_0^x y^2(t)dt, y(0)=1 \qquad (20)$$

With the exact solution $y(x)=e^x$

$y_0(x)$ are chosen as:

$$y_0(x)=\sum_{m=0}^{5}a_mx^m=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5,\ f(x)=1+x+\frac{x^2}{2!}+\frac{1}{2}-\frac{1}{2}(1+2x+2x^2)$$

In view of equation (5) we have,

$$y_1(x)=-a_0-a_1x-a_2x^2-a_3x^3-a_4x^4-a_5x^5+g(x)+\int_0^x\left(\int_0^x A_0(t)dt\right)dx=0$$

Here, A_0 is the Adomian polynomial. Now, $a_m, m=0,1$ are found from the relation:

$$y_1(x)=-a_0-a_1x-a_2x^2-a_3x^3-a_4x^4-a_5x^5+x-\frac{1}{6}x^3-\frac{1}{8}x^4-\frac{7}{120}x^5-\frac{31}{720}x^6$$

$$+\int_0^x\left(\int_0^x\left(a_0+a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5\right)^2dt\right)dx=0$$

which gives:

$$a_0=1, a_1=1, a_2=\frac{1}{2}, a_3=\frac{1}{6}, a_4=\frac{1}{24}, a_5=\frac{1}{120}$$

So, the solution will be $y(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}$

Which is the same with the exact solution.

So, we can increase the number of term to ∞, i.e.

$$y_0(x)=\sum_{m=0}^{\infty}a_mx^m$$

$$y(x)=-\sum_{m=0}^{\infty}a_mx^m+x-\frac{1}{6}x^3-\frac{1}{8}x^4-\frac{1}{120}x^5+\int_0^x\left(\left(\int_0^x\sum_{0}^{\infty}a_mt^m\right)dt\right)dx$$

This is the same with the exact solution:

$$y(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

Conclusions

The aim of this paper is to use the new modification of Adomian decomposition method to get the exact solution for the linear and nonlinear integro-differential equations which was achieved.

So far, with four different numerical examples, we have discussed the durability and effectiveness of the method.

Some problems were solved to demonstrate the usability of the new modification of Adomian decomposition method. It was discovered that the new modification of Adomian decomposition method work very well and faster after taking few iterations. The method needs much less computational work compared with traditional methods.

Recommendations

The modified Adomian decomposition method presented in this research proved to be efficient. The solution obtained converged rapidly to the exact solutions of the numerical problems considered. The method is therefore strongly recommended as a tool for the solution of strongly nonlinear integro-differential equations.

References

Kamel A.and Fathi A. (2005). Decomposition method for solving nonlinear integro-differential equations. *Journal of Applied mathematics and computing*. Vol. 19, No 1-2, 415-425.

Anjan B. (2000). Integro-differential perturbation of optical solutions. *Journal of Optics A*. Vol.2, No 5, 380-388.

Anjan B. (2003). Integro-differential perturbation of dispersion- managed solutions. *Journal of Electromagnetic waves and Applications*. Vol. 17, No 4, 641-665.

Zhao, J., and Corless, R.M. (2006). Compact finite difference method for integro-differential, *Appl. math. comput.*, 177, 271-288.

Abbasbandy, S., and Taati, A. (2003). Numerical solution of the system of nonlinear Volterra integro-differential equations with nonlinear differential part by the operational Tau method and error estimation, *Journal of comput Appl. math*. 231, 106-113.

Sekar, S., and Jaisankar, C. (2013). Numerical solution for the integro-differential equations using single term Haar wavelet series techniques. *International Journal of mathematical Archive*, 4, 97-103.

Biazar, J., Ghazvini, H., and Eslami, M. (2009). He's homotopy perturbation method for systems of integro-differential equations, *Chaos, Solitons and Fractals*, 39. 1253- 1258.

Biazar, J. and Eslami, M. (2011). Modified HPM for solving systems of Volterra integral equations of the second kind. *Journal of King Saud University-Science,* 23(1), 35-39.

Eslami, M. (2014). New Homotopy perturbation method for a special kind of Volterra integral equations in two-dimensional space, *Computational Mathematical and Modeling*, 25(1), 135-148.

Biazar, J. and Eslami, M (2011). A new homotopy perturbation method for solving systems of partial differential equations, *Computers and Mathematics with Applications* 62 (1), 225-234.

Biazar, J., Eslami, M. and Aninikhah, H. (2009). Application of homotopy perturbation method for systems of Volterra integral equations of the first kind. Chaos, *Solition and Fractals,* 42 (5), 3020-3026.

Eslami, M. and Mirzazadehi, M. (2014). Study of convergence of homotopy perturbation method for two-dimensional linear Volterra integral equations of the first kind. *Int. J. computing science and mathematics* 5(1), 72-80.

Maleknejad, K., Basirat, B. and Hashemizadeh, E. (2011). Hybrid legendre polynomials and Block-pulse functions approach for nonlinear volterra fredholin integro-differential equations. *Comput. Maths. Appl.,* 61, 2821-2828.

Wazwaz, A.M. (2011). Linear and Nonlinear Integral Equations Methods and Apllications, Springer, 1st Edition 2011.

Adomian, G. (1988). A review of the decomposition method in applied mathematics. *J. math. Anal. Appl*. 135,501-544.

Adomian, G. (1991). Solving frontier problems modeled by nonlinear partial differential equation. *Comp. Maths. Appl*., 22(8), 91-94.

Wazwaz, A.M. (2001). A reliable algorithm for solving boundary value problems for higher order integro-differential equations. *Applied Mathematical and Computation*, 118, 327-342.

Olayiwola, M.O., Ozoh, P., Usman, M.A. and A.W. Gbolagade. (2018). On the Numerical Solution of Linear and Non-Linear Fredholm Integral Equations. *Nigerian Journal of Mathematics and Applications,* Vol. 27, 2018, 68-75.

Rabbani, M., and Zarali, B. (2012): Solution of Fredholm integro-Differential Equations System of Modified Decomposition Method. *The Journal of Mathematics and Computer Science,* vol. 5, No 4, 258-264.

Hendi, F.A. and Bakodah, H.O. (2012). Numerical solution of Fredholm Volterra integral equations two-dimensional space by using discrete Adomian decomposition method. *IJRRAS*, 10(3), 466-471.

Manafianheris, J. (2012). Solving the integro-differential equations using the modified Laplace Adomian Decomposition method, *Journal of Mathematical Extension*, vol. 6, No 1, 41-55.

Alao, S., Akinboro, F.S. and Akinpelu F.O., and Oderinu, R.A. (2014). Numeriacal Solution of Integro-Differential Equation using Adomian Decomposition and Variational Iteration Methods. *IOSR Journal of Mathematics (IOSR-JM)*, vol. 10, Issue 4 Ver.II, 18-22.

Hossein Jafari, Tayyebi E., Sadeghi, S. and Khalique, C.M. (2014). A new modification of the Adomian decomposition method for nonlinear integral equations. *Int. J. Adv. Appl. Maths. and Mech.,* 1(4), 33-39.

Bakodah, H.O., Al-mazmumy, M. and Almuhalbedi, S.O. (2017). Efficient modification of the Adomian decomposition method for solving integro-differential equations. An Inter. *Journal Maths. Sci.*, No 1, 15-21.