Dynamic Response of a Slender Member Under Moving Loads with Time-Dependent Boundary Conditions

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Abstract

A method known as Mindlin Goodman's was implored in transforming our inhomogeneous governing equation cum inhomogeneous bcs and Ics to homogeneous equations and Generalized Fourier integral transformation method were used to treat our dynamical problem. The moving force case of the system was first investigated for the dynamical response of the slender member under the action of moving load(s) leading to a closed form solution. More so, vibration of variable magnitude load moving with constant speed is observed for shear moduli, variable foundation, and axial force. The computed results of the structural parameters plotted in graphs reveals vividly visible effects of the stability of our dynamical system. Also established are the conditions by which our dynamical system experienced resonance phenomenon which is very essential in life and practical application or reality.

Keywords: Dynamic Response; moving force; moving mass; dynamical system.

Introduction

In calculating the vibrations of distributed parameter system carrying several travelling sub-systems cannot be overemphasized in the fields of Engineering and applied Mathematics as applicable. In general, we emphasis more on the dynamics of the distributed parameter systems rather than the moving loads: moving mass and moving force problems. In our system being considered example of distributed parameter system is a beam where the travelling sub-systems are moving cars or bicycles. A distributed parameter system is either elastic or visco-elastic. However, there could be elastic or and visco-elastic structural configurations on which one or more subsystems can travel. At this juncture, we remarked that stationery subsystems produce stresses and deformation that are constant while travelling subsystems produce effects which are variable functions of the position of the sub-system which is time bound that is function of time. If structural members are under the passage of moving loads, the reaction of the moving load and the structure makes the dynamic response calculation very cumbersome Fryba (1972). By virtue of the relevance in the analysis and design of distributed parameter systems, the dynamic response of structural members with their moving loads has been extensively researched into in the past and a number of experimental and numerical investigations have been sighted in literatures also in recent times Ajibola (2017), Oni Sunday Tunbosun *et al.* (2012), Omolofe (2017), Ajibola (2014) and Gbadeyan and Agboola.(2012): In this research on the effects of cars moving over large-span bridges, Inglis. (1934.). Iintroduced a theory according to where the gravitational effects of the moving sub system may be separated from the inertia ones. In the calculation, the force is considered as moving along the beam while the mass of the vehicle acts at a definite, constant point, say x_0. The argument has been that the second part of the assumption is indeed justified or not justified. The inertia action of the mass in the deformed structure is described by the D'Alembert's principle as the product of mass and acceleration. If the inertia effect of the moving subsystem is considered, the governing equations of motion become complex and cumbersome but no longer possess coefficients that are constant. Hence, variable and singular coefficients evolved. But if the inertia effect of the moving sub system is neglected, the problem is referred to as moving force otherwise it is referred to as moving mass model. Although, moving force models has received great attention evident in literature; now the question arises: what is the reliability of any design considering this assumption? This assumption would be justified if it can be established that the solution of the approximated model has been proved to be an upper bound for the actual deflection of the elastic system. But results in most literatures have shown that it is not so. Thus, approximate model in which the Vehicle-track interaction is completely neglected has been described by Giuseppe Muscolino and Alessandro Palmori (2007) as the crudest approximation known to the literature of assessing the dynamic response of an elastic system which supports moving concentrated masses.

Mathematical Formulation

The problem of the vibration of a Uniform Rayleigh beam model acted upon by moving concentrated subsystems $P(x,t)$ was considered. The transverse displacement $U(x,t)$, of a Uniform Rayleigh beam being transverse by a mass (load) M traveling at a uniform velocity u which as a Length L is governed by the fourth partial differential equation thus,

$$\frac{\partial^2}{\partial x^2}\left[\frac{EI\,\partial^2 U(x,t)}{\partial x^2}\right]-\frac{N\partial^2 U(x,t)}{\partial x^2}+KU(x,t)-\mu r^2\frac{\partial^4 U(x,t)}{\partial x^2\partial t^2}+\frac{\mu\partial^2 U(x,t)}{\partial t^2}=P_f(x,t)\left[1-\frac{1}{g}\frac{d^2U(x,t)}{dt^2}\right]$$

1.

x is the spatial co-ordinate, t is the time, EI is the flexural rigidity of the beam, μ is the mass / unit length of the beam, r is the radius of gyration, N is the axial force and K is the elastic foundation. The continuous moving force $P_f(x,t)$ acting on the beam model is given by $P_f(x,t)=Mg\delta(x-f(t))$

2.

The time t is assumed to be limited to that interval of time within which the mass M acts on the beam i.e.

$$0\le ut\le L \text{ and } \delta(x-ut) \qquad 3.$$

is the Dirac – delta function defined as:

$$\delta(x-ut)=\begin{cases}0 & x\neq ut\\ \infty & x=ut\end{cases} \qquad 4.$$

Now, on substituting equations 2 to 4 into 1 and assuming that the EI and μ, do not vary with x along the span L .Hence, equation 1 results to

$$EI\frac{\partial^4 U(x,t)}{\partial x^4}-\frac{N\partial^2 U(x,t)}{\partial x^2}+\frac{\mu\,\partial^2 U(x,t)}{\partial t^2}+KU(x,t)-\mu r^2\frac{\partial^4 U(x,t)}{\partial x^2\partial t^2}$$
$$=Mg\delta(x-ut)\left[1-\frac{1}{g}\left(\frac{\partial^2 U(x,t)}{\partial t^2}+\frac{2u\partial^2 u(x,t)}{\partial x\partial t}+\frac{u^2\partial^2 u(x,t)}{\partial x^2}\right)\right] \qquad 5.$$

The boundary conditions (bcs) of this problem are taken to be time dependent which means that at each of the boundary points, there are two bcs written as:

$$D_i\left[U(0,t)\right]=F_i(t) \quad i=1,2 \text{ and } D_i\left[U(L,t)\right]=F_i(t) \quad i=3,4 \qquad 6$$

The initial conditions (Ics) of the motion at time $t=0$ may in general be specified by two arbitrary functions thus; $U(x,0)=U_0(x)$ and $\frac{\partial U(x,0)}{\partial t}=\dot{U}_0(x)$ 7.

Operational Simplification of Equation

The analytical solution to our dynamical system with bcs and Ics 1, 6 and 7 is sought with the use of an approach due to Mindlin and Goodman (1950). is extended to obtain a robust technique capable of handling these class of problems for all variants of conditions.

First, an auxiliary variable $z(x,t)$ in the form [5]

$$U(x,t)=Z(x,t)+\sum_{i=1}^{4}F_i(t)g_i(x) \qquad 8.$$

Was implored having substituted equation 6 into 7 transformed our bvp in terms of $U(x,t)$ into the bvp in terms of $Z(x,t)$. The functions $g_i(x)$ are referred to as the displacement influences functions while that of $f_i(t)$ are pertinent displacements at every boundary. The functions $g_i(x)$ are chosen to render the bcs of our bvp in $Z(x,t)$ homogeneous.

Hence, in view of equations 6, 7 and 8 into equation 5 one obtains

$$\frac{EI}{\mu}\frac{\partial^4 Z(x,t)}{\partial x^4}-\frac{N}{\mu}\frac{\partial^2}{\partial x^2}Z(x,t)+\frac{\partial^2}{\partial t^2}Z(x,t)+\frac{K}{\mu}Z(x,t)$$

$$
\begin{aligned}
&-\frac{r^2\partial^4 Z(x,t)}{\partial x^2\partial t^2}+\frac{M}{\mu}\delta(x-ut)\left[\frac{\partial^2 Z(x,t)}{\partial t^2}+\frac{2U\partial^2 Z(x,t)}{\partial x\partial t}+\frac{U^2\partial^2 Z(x,t)}{\partial x^2}\right]\\
&=\frac{M}{\mu}g\delta(x-ut)-\frac{EI}{\mu}\sum_{i=1}^{4} f_i(t)g_i'''(x)+\frac{N}{\mu}\sum_{i=1}^{4} f_i(t)g_i(x)\\
&-\sum_{i=1}^{4}\ddot{f}_i(t)g_i(x)-\frac{K}{\mu}\sum_{i=1}^{4} f_i(t)g_i(x)+r^2\sum_{i=1}^{4} f_i''(t)g_i''(x)\\
&+\frac{M}{\mu}\delta(x-ut)\left[\sum_{i=1}^{4}\left(\ddot{f}_i(x)g_i(x)+2U\dot{f}_i(x)g_i'(x)+U^2 f_i(t)g_i''(x)\right)\right] \qquad 9.
\end{aligned}
$$

Solution Procedure

It is observed that equation 9 is a fourth order pde having some coefficients which are not only variable but are also singular. These coefficients are the Dirac delta functions which multiply each term of the convective acceleration operator associated with the inertia of the masses of the traversing sub systems. It is remarked here that the new transformed equation is now amenable to the robust method of generalized finite integral transform proposed by Oni (2012).

The Generalized Finite Integral Transform Method

Generalized finite integral transform method is about the best method that can be adopted to handle problems involving mechanical vibrations. This integral transform method is given by

$$\bar{z}(m,t)=\int_o^l z(x,t)V_m(x)dx \quad \text{With the inverse} \quad z(x,t)=\sum_{m=1}^{\infty}\frac{\mu}{V_m}\bar{z}(m,t)V_m(x) \qquad 10$$

$\bar{V}_m=\int_o^l \mu V_m^2(x)dx$ and $V(x,t)$ is any function such that $f_i(t)$ are satisfied. An appropriate selection of functions for any problem involving beams are beam mode shape. Thus the m^{th} normal mode of deflections of a uniform Rayleigh beam given by

$$V_m(x)=Sin\frac{\lambda_m x}{L}+A_m Cos\frac{\lambda_m x}{L}+B_m Sinh\frac{\lambda_m x}{L}+C_m Cosh\frac{\lambda_m x}{L} \qquad 11$$

is chosen as a suitable kernel of the integral where (λ_m) is the mode frequency where A_m, B_m and C_m are constants. An important feature of the use of this kernel is that it makes the transformation suitable for all variants of bcs of our dynamical problem. The parameters λ_m, A_m, B_m and C_m are obtained when the equation 11 is substituted into the appropriate boundary conditions. By applying 10, equation 9 takes the form

$$
\begin{aligned}
&\bar{Z}_{tt}(m,t)=B_1Q_A(t)+B_2Q_B(t)+B_3Z(m,t)+B_1Z(0,L,t)-r^2Q_C(t)+Q_D(t)+Q_E(t)+Q_F(t)\\
&PV_m(Ut)-\left[G_a(t)-G_b(t)+G_c(t)+G_d(t)+G_e(t)+G_f(t)+G_g(t)+G_h(t)\right] \qquad 12
\end{aligned}
$$

where $B_1=\frac{EI}{\mu},\ B_2=\frac{N}{\mu},\ B_3=\frac{K}{\mu},\ P=\frac{mg}{\mu},$ and $\varepsilon=\frac{M}{\mu L}$ 13

$$Q_A(t)=\int_o^l \frac{\partial^4}{\partial x^4}Z(x,t)V_m(x)dx\ , \qquad Q_B(t)=\int_o^l \frac{\partial^2}{\partial x^2}Z(x,t)V_m(x)dx$$

$$Q_C(t)=\int_o^l \frac{\partial^4}{\partial x^2\partial t^2}Z(x,t)V_m(x)dx,\ Q_D(t)=\int_o^l \frac{M}{\mu}\delta(x-ut)\frac{\partial^2}{\partial t^2}Z(x,t)V_m(x)dx$$

$$Q_E(t)=\int_o^l \frac{2MU}{\mu}\delta(x-ut)\frac{\partial^2}{\partial x\partial t}Z(x,t)V_m(x)dx$$

$$Q_F(t)=\int_o^l \frac{M}{\mu}U^2\delta(x-ut)\frac{\partial^2}{\partial x^2}Z(x,t)V_m(x)dx \qquad \mathbf{14}$$

$$Z(0,L,t)=\left[V_m(x)\frac{\partial^3}{\partial x^3}Z(x,t)-V_m'(x)\frac{\partial^2}{\partial x^2}Z(x,t)+V_m''(x)\frac{\partial}{\partial x}Z(x,t)-V_m'''(x)Z(x,t)\right]_0^L \qquad 15$$

$$G_a(t)=B_1\sum_{i=1}^4 f_i(t)\int_0^L\left(\frac{d^4}{dx^4}g_i(x)\right)V_m(x)dx,\ G_b(t)=B_2\sum_{i=1}^4 f_i(t)\int_0^L\left(\frac{d^2}{dx^2}g_i(x)\right)V_m(x)dx \qquad 16$$

$$G_c(t)=\sum_{i=1}^4\ddot{f}_i(t)\int_0^L g_i(x)V_m(x)dx,\ G_d(t)=B_3\sum_{i=1}^4\ddot{f}_i(t)\int_0^L g_i(x)V_m(x)dx \qquad 17$$

$$G_e(t)=r^2\sum_{i=1}^4\ddot{f}_i(t)\int_0^L\left(\frac{d^2}{dx^2}g_i(x)\right)V_m(x)dx, G_f(t)=\frac{M}{\mu}\sum_{i=1}^4\ddot{f}_i(t)\int_0^L\delta(x-ut)g_i(x)V_m(x)dx \qquad 18$$

$$G_g(t)=\frac{2MU}{\mu}\sum_{i=1}^4\dot{f}_i(t)\int_0^L\delta(x-ut)g_i'(x)V_m(x)dx\ G_h(t)=\frac{MU^2}{\mu}\sum_{i=1}^4 f_i(t)\int_0^L\delta(x-ut)g_i''(x)V_m(x)dx$$

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It is obvious that 11 satisfy the homogeneous forth order DE for the Euler beam. Thus

$$EI\frac{d^4}{dx^4}V_m(x)-\mu\omega_m^2V_m(x)=0 \qquad 20$$

The parameter (ω) is the natural circular frequency defined by $\omega_m^2=\frac{\lambda^4}{L^4}\frac{EI}{\mu}$

In view of equation 20, equation 12 become

$$EI\int_o^L\left(\frac{d^4}{dx^4}V_m(x)\right)Z(x,t)dx=\mu\omega_m^2\int_0^L V_m(x)Z(x,t)dx \qquad 21$$

Thus, $Q_A(t)=\frac{\mu}{EI}\omega_m^2\overline{Z}(m,t)$

Evaluating the evolving 42 integrals in equations 14, after some substitutions, simplifications and arrangements equation 12 yields

$$\begin{aligned}
&\overline{Z}_{tt}(m,t)+\left(\omega_m^2+\frac{K}{\mu}\right)\overline{Z}_t(m,t)-\frac{N}{\mu}\sum_{k=1}^{\infty}\overline{Z}(k,t)S_1^*(k,m)-r^2\sum_{k=1}^{\infty}\overline{Z}_{tt}(k,t)S_1^*(k,m)+\\
&+\frac{2M}{\mu L}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}Cos\frac{n\pi ut}{L}\overline{Z}_{tt}(k,t)S_{2c}^*(k,m,n)+\frac{2MU}{\mu L}\sum_{k=1}^{\infty}\overline{Z}(k,t)S_3^*(k,m)\\
&\quad+\frac{4MU}{\mu L}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}Cos\frac{n\pi ut}{L}\overline{Z}_t(k,t)S_{3c}^*(k,m,n)+\frac{MU^2}{\mu L}\sum_{k=1}^{\infty}\overline{Z}(k,t)S_1^*(k,m)\\
&\quad+\frac{2MU^2}{\mu L}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}Cos\frac{n\pi ut}{L}\overline{Z}_t(k,t)S_{1c}^*(k,m,n)+\frac{M}{\mu L}\sum_{k=1}^{\infty}\overline{Z}_{tt}(k,t)S_2^*(k,m)
\end{aligned}$$

$$= PV_m(U,t) - \left[G_a(t) - G_b(t) + G_c(t) + G_d(t) - G_e(t) + G_f(t) + G_g(t) + G_h(t)\right] \qquad 23$$

Where $S_1^*(k,m) = \frac{\mu}{V_k}\int_0^L V_k''(x)V_m(x)dx$, $S_{1c}^*(k,m,n) = \frac{\mu}{V_k}\int_0^L Cos\frac{n\pi x}{L}V_k''(x)V_m(x)dx$

$$S_2^*(k,m) = \frac{\mu}{V_k}\int_0^L V_k(x)V_m(x)dx, S_{2c}^*(k,m) = \frac{\mu}{V_k}\int_0^L Cos\frac{n\pi x}{L}V_k(x)V_m(x)dx$$

$$S_3^*(k,m) = \frac{\mu}{V_k}\int_0^L V_k'(x)V_m(x)dx, S_{3c}^*(k,m) = \frac{\mu}{V_k}\int_0^L Cos\frac{n\pi x}{L}V_k'(x)V_m(x)dx \qquad 24$$

Using 14 to 18 and its derivatives in integrals 24 to obtain

$$S_1^*(k,m) = -\frac{\lambda_k^2}{k_0(x)L^2}\begin{bmatrix} I_1 + A_m I_2 + B_m I_3 + C_m I_4 + A_k I_5 + A_k A_m I_6 + A_k B_m I_7 + A_k C_m I_8 + B_k I_9 \\ -B_k A_m I_{10} - B_k B_m I_{11} - B_k C_m I_{12} + C_k I_{13} - C_k A_m I_{14} - C_k B_m I_{15} - C_k C_m I_{16} \end{bmatrix}$$

$$S_2^*(k,m) = \frac{1}{k_0(x)}\begin{bmatrix} I_1 + A_m I_2 + B_m I_3 + C_m I_4 + A_k I_5 + A_k A_m I_6 + A_k B_m I_7 + A_k C_m I_8 + B_k I_9 \\ +B_k A_m I_{10} + B_k B_m I_{11} + B_k C_m I_{12} + C_k I_{13} + C_k A_m I_{14} + C_k B_m I_{15} + C_k C_m I_{16} \end{bmatrix}$$

$$S_3^*(k,m) = \frac{\lambda^2}{k_0(x)L}\begin{bmatrix} -A_k I_1 - A_k A_m I_2 - A_k B_m I_3 - A_k C_m I_4 + I_5 + A_m I_6 + B_m I_7 + C_m I_8 + C_k I_9 \\ +C_k A_m I_{10} + C_k B_m I_{11} + C_k C_m I_{12} + B_k I_{13} + B_k A_m I_{14} + B_k B_m I_{15} + B_k C_m I_{16} \end{bmatrix}$$

$$S_{1c}^*(k,m) = \frac{\lambda_k^2}{k_0(x)L^2}\begin{bmatrix} -I_7 - A_m I_{18} - B_m I_{19} - C_m I_{20} - A_k I_{21} - A_k A_m I_{22} - A_k B_m I_{23} - A_k C_m I_{24} - B_k I_{25} \\ +B_k A_m I_{26} + B_k B_m I_{27} + B_k C_m I_{28} + C_k I_{29} + C_k A_m I_{30} + C_k B_m I_{31} + C_k C_m I_{32} \end{bmatrix}$$

$$S_{2c}^*(k,m) = \frac{1}{k_0(x)}\begin{bmatrix} I_{17} + A_m I_{18} + B_m I_{19} + C_m I_{20} + A_k I_{21} + A_k A_m I_{22} + A_k B_m I_{23} + A_k C_m I_{24} + B_k I_{25} \\ +B_k A_m I_{26} + B_k B_m I_{27} + B_k C_m I_{28} + C_k I_{29} + C_k A_m I_{30} + C_k B_m I_{31} + C_k C_m I_{32} \end{bmatrix}$$

$$S_{3c}^*(k,m) = \frac{\lambda^2}{k_0(x)L}\begin{bmatrix} -A_k I_{17} - A_k A_m I_{18} - A_k B_m I_{19} - A_k C_m I_{20} + I_{21} + A_m I_{22} + B_m I_{23} + C_m I_{24} + C_k I_{25} \\ +C_k A_m I_{26} + C_k B_m I_{27} + C_k C_m I_{28} + B_k I_{29} + B_k A_m I_{30} + B_k B_m I_{31} + B_k C_m I_{32} \end{bmatrix}$$

$$k_0(x) = \int_0^l V_k^2(x)dx$$

$$k_0(x) = \left[I_{33} + 2A_k I_{34} + 2B_k I_{35} + 2C_k I_{36} + A_k^2 I_{37} + 2A_k B_k I_{38} + 2A_k C_k I_{39} + B_k^2 I_{40} + 2B_k C_k I_{41} + C_k^2 I_{42}\right] \qquad 25$$

where

$$I_1 = \int_0^l \sin\frac{\lambda_k x}{L}\sin\frac{\lambda_m x}{L}dx,\ I_2 = \int_0^l \sin\frac{\lambda_k x}{L}\cos\frac{\lambda_m x}{L}dx,\ I_3 = \int_0^l \sin\frac{\lambda_k x}{L}\sinh\frac{\lambda_m x}{L}dx$$

$$I_4 = \int_0^l \sin\frac{\lambda_k x}{L}\cosh\frac{\lambda_m x}{L}dx,\ I_5 = \int_0^l \cos\frac{\lambda_k x}{L}\sin\frac{\lambda_m x}{L}dx,\ I_6 = \int_0^l \cos\frac{\lambda_k x}{L}\cos\frac{\lambda_m x}{L}dx$$

$$I_7 = \int_0^l \cos\frac{\lambda_k x}{L}\sinh\frac{\lambda_m x}{L}dx,\ I_8 = \int_0^l \cos\frac{\lambda_k x}{L}\cosh\frac{\lambda_m x}{L}dx,\ I_9 = \int_0^l \sinh\frac{\lambda_k x}{L}\sin\frac{\lambda_m x}{L}dx$$

$$I_{10} = \int_0^l \sinh\frac{\lambda_k x}{L}\cos\frac{\lambda_m x}{L}dx,\ I_{11} = \int_0^l \sinh\frac{\lambda_k x}{L}\sinh\frac{\lambda_m x}{L}dx,\ I_{12} = \int_0^l \sinh\frac{\lambda_k x}{L}\cosh\frac{\lambda_m x}{L}dx$$

$$I_{13}=\int_0^l \cosh\frac{\lambda_k x}{L}\sin\frac{\lambda_m x}{L}dx,\ I_{14}=\int_0^l \cosh\frac{\lambda_k x}{L}\cos\frac{\lambda_m x}{L}dx,\ I_{15}=\int_0^l \cosh\frac{\lambda_k x}{L}\sinh\frac{\lambda_m x}{L}dx$$

$$I_{16}=\int_0^l \cosh\frac{\lambda_k x}{L}\cosh\frac{\lambda_m x}{L}dx,\ I_{17}=\int_0^l \cos\frac{n\pi x}{L}\sin\frac{\lambda_k x}{L}\sin\frac{\lambda_m x}{L}dx,\ I_{18}=\int_0^l \cos\frac{n\pi x}{L}\sin\frac{\lambda_k x}{L}\cos\frac{\lambda_m x}{L}dx$$

$$I_{19}=\int_0^l \cos\frac{n\pi x}{L}\sin\frac{\lambda_k x}{L}\sinh\frac{\lambda_m x}{L}dx,\ I_{20}=\int_0^l \cos\frac{n\pi x}{L}\sin\frac{\lambda_k x}{L}\cosh\frac{\lambda_m x}{L}dx$$

$$I_{21}=\int_0^l \cos\frac{n\pi x}{L}\cos\frac{\lambda_k x}{L}\sin\frac{\lambda_m x}{L}dx,\ I_{22}=\int_0^l \cos\frac{n\pi x}{L}\cos\frac{\lambda_k x}{L}\cos\frac{\lambda_m x}{L}dx$$

$$I_{23}=\int_0^l \cos\frac{n\pi x}{L}\cos\frac{\lambda_k x}{L}\sinh\frac{\lambda_m x}{L}dx,\ I_{24}=\int_0^l \cos\frac{n\pi x}{L}\cos\frac{\lambda_k x}{L}\cosh\frac{\lambda_m x}{L}dx$$

$$I_{25}=\int_0^l \cos\frac{n\pi x}{L}\sinh\frac{\lambda_k x}{L}\sin\frac{\lambda_m x}{L}dx,\ I_{26}=\int_0^l \cos\frac{n\pi x}{L}\sinh\frac{\lambda_k x}{L}\cos\frac{\lambda_m x}{L}dx$$

$$I_{27}=\int_0^l \cos\frac{n\pi x}{L}\sinh\frac{\lambda_k x}{L}\sinh\frac{\lambda_m x}{L}dx,\ I_{28}=\int_0^l \cos\frac{n\pi x}{L}\sinh\frac{\lambda_k x}{L}\cosh\frac{\lambda_m x}{L}dx$$

$$I_{29}=\int_0^l \cos\frac{n\pi x}{L}\cosh\frac{\lambda_k x}{L}\sin\frac{\lambda_m x}{L}dx,\ I_{30}=\int_0^l \cos\frac{n\pi x}{L}\cosh\frac{\lambda_k x}{L}\cos\frac{\lambda_m x}{L}dx$$

$$I_{31}=\int_0^l \cos\frac{n\pi x}{L}\cosh\frac{\lambda_k x}{L}\sin\frac{\lambda_m x}{L}dx,\ I_{32}=\int_0^l \cos\frac{n\pi x}{L}\cosh\frac{\lambda_k x}{L}\cosh\frac{\lambda_m x}{L}dx$$

$$I_{33}=\int_o^l \sin^2\frac{\lambda_k x}{L}dx,\ I_{34}=\int_o^l \sin\frac{\lambda_k x}{L}\cos\frac{\lambda_k x}{L}dx\quad I_{35}=\int_o^l \sin\frac{\lambda_k x}{L}\sinh\frac{\lambda_k x}{L}dx$$

$$I_{36}=\int_o^l \sin\frac{\lambda_k x}{L}\cosh\frac{\lambda_k x}{L}dx\ I_{37}=\int_o^l \cos^2\frac{\lambda_k x}{L}dx,\quad I_{38}=\int_o^l \cos\frac{\lambda_k x}{L}\sinh\frac{\lambda_k x}{L}dx$$

$$I_{39}=\int_o^l \cos\frac{\lambda_k x}{L}\cosh\frac{\lambda_k x}{L}dx,\quad I_{40}=\int_o^l \sinh\frac{\lambda_k x}{L}dx,\ I_{41}=\int_o^l \sinh\frac{\lambda_k x}{L}\cosh\frac{\lambda_k x}{L}dx,$$

$I_{42}=\int_o^l \cosh^2\frac{\lambda_k x}{L}dx$. are the involving integrals 26

Using the solutions of the involving integrals 26 in 23 to have

$$\begin{aligned}
&\bar{Z}_{tt}(m,t)+\alpha_m^2\bar{Z}_t(m,t)-\frac{N}{\mu}\sum_{k=1}^{\infty}\bar{Z}(k,t)S_1^*(k,m)-r^2\sum_{k=1}^{\infty}\bar{Z}_{tt}(k,t)S_1^*(k,m)+\varepsilon\Bigg[\sum_{k=1}^{\infty}\bar{Z}_{tt}(k,t)S_2^*(k,m)\\
&+2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}Cos\frac{n\pi ut}{L}\bar{Z}_{tt}(k,t)S_{2c}^*(k,m,n)+2U\sum_{k=1}^{\infty}\bar{Z}(k,t)S_3^*(k,m)\\
&+4U\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}Cos\frac{n\pi ut}{L}\bar{Z}_t(k,t)S_{3c}^*(k,m,n)+U^2\sum_{k=1}^{\infty}\bar{Z}(k,t)S_1^*(k,m)\\
&+2U^2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}Cos\frac{n\pi ut}{L}\bar{Z}_t(k,t)S_{1c}^*(k,m,n)\\
&=P\left[Sin\frac{\lambda_m ut}{L}+A_m Cos\frac{\lambda_m ut}{L}+B_m Sinh\frac{\lambda_m ut}{L}+C_m Cosh\frac{\lambda_m ut}{L}\right]\\
&-\left[G_a(t)-G_b(t)+G_c(t)+G_d(t)-G_e(t)+G_f(t)+G_g(t)+G_h(t)\right]
\end{aligned}\qquad 27$$

where G_a, G_b, G_c ....G_h are as defined in equations 15-18

$$\alpha_m^2=\left(\omega_m^2+\frac{k}{\mu}\right)\qquad 28$$

Equation 27 is the evolving coupled transformed second order DE representing the problem of response of uniform Rayleigh beams under concentrated moving masses for all variants of bcs. In this case, the analyses were presented with examples of common bcs which are mostly encountered in Engineering practice; it is worth to mention that the G's are not evaluated at this point because the G's are condition dependent.

Simply Supported Boundary Conditions

At this juncture, the uniform Rayleigh beam has simple support at ends x=0 and x=L, the bending moment and displacement vanished at the simply supported ends hence, the conditions are expressed as:

$$\bar{Z}(0,t)=0=\bar{Z}(L,t) \quad , \qquad \frac{d^2 Z(0,t)}{dx^2}=0=\frac{d^2\bar{Z}(l,t)}{dx^2} \tag{29}$$

Hence, for normal modes

$$\bar{V}_m(0)=0=\bar{V}_m(L) \quad , \qquad \frac{d^2\bar{V}_m(0)}{dx^2}=0=\frac{d^2 V_m(L)}{dx^2} \tag{30}$$

which implies that

$$\bar{V}_k(0)=0=\bar{V}_k(L) \quad , \qquad \frac{d^2\bar{V}_k(0)}{dx^2}=0=\frac{d^2 V_k(L)}{dx^2} \tag{31}$$

Thus, making use of equations 29 - 31 into the beam function, it can be shown that

$$A_m=0, B_m=0, C_m=0 \quad \text{and} \quad A_k=0, B_k=0, C_k=0 \tag{32}$$

And the frequency equation becomes:

$$Sin\lambda_m = Sin\lambda_k = 0 \quad \text{implies} \quad \lambda_m = m\pi \quad \text{and} \quad \lambda_k = k\pi \tag{33}$$

Substituting equations 32 into the transformed equation 27 one obtains the transformed equation for Rayleigh beam resting on elastic foundations, having simple supports at both edges, that is;

$$\begin{aligned}
&\bar{Z}_{tt}(m,t)+\left(\frac{m^4\pi^4}{L^4}\frac{EI}{\mu}+\frac{k}{\mu}\right)\bar{Z}(m,t)-\sum_{k=1}^{\infty}\left(-\frac{k^2\pi^2}{k_0(x)L^2}\right)\left[r^2\bar{Z}_{tt}(k,t)I_1^*+\frac{N}{\mu}\bar{Z}(k,t)I_1^*\right] \\
&+\varepsilon\left[\frac{1}{k_0(x)}\left(\left[\bar{Z}_{tt}(k,t)\right]I_1^*+2\sum_{nk=1}^{\infty}Cos\frac{n\pi ut}{L}I_{17}^*\right)+\frac{2Uk\pi}{Lk_0(x)}\left(I_5^*+2\sum_{nk=1}^{\infty}Cos\frac{n\pi ut}{L}I_{21}^*\right)\bar{Z}_t(k,t)\right) \\
&+\frac{U^2k^2\pi^2}{L^2k_0(x)}\left(-I_1^*+2\sum_{nk=1}^{\infty}Cos\frac{n\pi ut}{L}(-I_{17}^*)\right)\bar{Z}(k,t)\Bigg]=PSin\frac{m\pi ut}{L}+ \\
&=-\left[G_a(t)-G_b(t)+G_c(t)+G_d(t)-G_e(t)+G_f(t)+G_g(t)+G_h(t)\right]
\end{aligned} \tag{34}$$

$$I_1^*=\int_0^l \sin\frac{k\pi x}{L}\sin\frac{m\pi x}{L}dx=\begin{cases}0, & if \quad m\neq k \\ \dfrac{L}{2} & if \quad m=k\end{cases} \tag{35}$$

$$k_0(x)=\int_0^l \sin^2\frac{k\pi x}{L}dx=\begin{cases}0, & if \quad m\neq k \\ \dfrac{L}{2} & if \quad m=k\end{cases} \tag{36}$$

$$I_5^*=\int_0^l \cos\frac{k\pi x}{L}\sin\frac{m\pi x}{L}dx=\begin{cases}0, & if \quad m\pm k=even \\ \dfrac{2Lm}{(m^2-k^2)\pi} & if \quad m\pm k=odd\end{cases} \tag{37}$$

$$I_{17}^*=\int_0^l \cos\frac{n\pi x}{L}\sin\frac{k\pi x}{L}\sin\frac{m\pi x}{L}dx, \tag{38}$$

$$I_{21}^*=\int_0^l \cos\frac{n\pi x}{L}\cos\frac{k\pi x}{L}\sin\frac{m\pi x}{L}dx=\frac{2Lm\pi^3\left[n^2+k_2-m^2\right]}{\pi^2\left[(n+k)^2-m^2\right]\left[(n-k)^2-m^2\right]} \tag{39}$$

In view of 35-39, after some rearrangements in 30, one obtains

$$\begin{aligned}&\left(1+\frac{r^2m^2\pi^2}{L^2}\right)\overline{Z}_{tt}(m,t)+\left(\frac{m^4\pi^4}{L^4}\frac{EI}{\mu}+\frac{N}{\mu}\frac{m^2\pi^2}{L^2}\right)\overline{Z}(m,t)+\varepsilon\Bigg[\overline{Z}_{tt}(m,t)-\frac{m^2u^2\pi^2}{L^2}\overline{Z}(m,t)\\&+\sum_{k=1}^{\infty}\Bigg\{2Sin\frac{m\pi ut}{L}Sin\frac{k\pi ut}{L}\overline{Z}_{tt}(k,t)+\frac{2k^2u^2\pi^2}{L^2}Sin\frac{m\pi ut}{L}Sin\frac{k\pi ut}{L}\overline{Z}(k,t)+\\&\left(\frac{8muk}{(m^2-k^2)}+\sum_{n=1}^{\infty}\frac{16muk\pi^2\left[n^2+k_2-m^2\right]}{\pi^2\left[(n+k)^2-m^2\right]\left[(n+k)^2-m^2\right]}Cos\frac{n\pi ut}{L}\right)\overline{Z}_t(k,t)\Bigg\}\Bigg]\\&=PSin\frac{m\pi ut}{L}-[G_a(t)-G_b(t)+G_c(t)+G_d(t)-G_e(t)+G_f(t)+G_g(t)+G_h(t)]\end{aligned}\tag{40}$$

Where $\varepsilon=\dfrac{M}{\mu L}$ 41

At this junction, it is pertinent to obtain the particular function $g_i(x)$ that ensures zeros of the right-hand sides of the boundary conditions for a Simply-supported beam. To this end, the function $g_i(x)$ is taken to be a third-degree polynomial, thus

$$g_i(x)=a_ix^3+b_ix^2+c_ix+d_i\,,\qquad i=1,2,3,4 \tag{42}$$

To obtain $g_i(x)$ explicitly, it is required to satisfy four conditions defined as

$$D_j\left[g_i(0)\right]=\delta_{ij,}\qquad j=1,2 \tag{43}$$

And

$$D_j\left[g_i(L)\right]=\delta_{ij,}\qquad j=3,4 \tag{44}$$

As such, a set of four algebraic equations involving the coefficients of $g_i(x)$ are obtained and then solved simultaneously to arrive at the values of the coefficients of $g_i(x)$ .Thus

$$\begin{aligned}&D_1\left[g_1(0)\right]=\delta_{11}=1,D_2\left[g_2(0)\right]=\delta_{12}=0\\&D_3\left[g_3(L)\right]=\delta_{13}=0,D_4\left[g_4(L)\right]=\delta_{14}=0\end{aligned}\tag{45}$$

Particularly, substituting $g_i(x)$ into equation 42 one obtains the following simultaneous equations

$$a_1=1\ ,2c_1=0\ ,a_1+b_1L+c_1L^2+d_1L^3=0 \text{ and } 2c_1+6d_1L=0 \tag{46}$$

Solving equations 46 simultaneously, one obtains

$$a_1=1\ ,b_1=-\tfrac{1}{L}\ ,c_1=0 \text{ and } d_1=0 \tag{47}$$

Hence, $g_1(x)=1-\frac{x}{L}$, similarly when $i=(2,3,4)$ we have:

$$g_2(x)=-\tfrac{L}{3}x+\tfrac{x}{2}+\tfrac{x^2}{6L}\ ,\quad g_3(x)=\tfrac{x}{L} \text{ and } g_4(x)=-\tfrac{L}{6}x+\tfrac{x^3}{6L} \tag{48}$$

It worth to mention here, that, we only computed those of the $g_i(x)$, for which the corresponding $f_i(t)$ do not turn to zero or vanish. In this investigation, we shall consider a simply supported beam, one of whose end $x=0$, (say) is subjected to a sine-wave undamped transient displacement, starting from rest and $x=L$ is subjected to a damped sine-wave transient displacement starting from rest. Thus, we can write

$$f_1=BSin\Omega t \text{ and } f_3=Ae^{-\beta t}Sin\Omega t \tag{49}$$

Where A, B are amplitudes, Ω is frequency and β is parameter.

Therefore, the required $g_i(x)$ are $g_1(x)$ and $g_3(x)$. The determination of $f_1(t)$, $f_3(t)$ $g_1(x)$ and $g_3(x)$ permitted the complete determination of the right hand side of the Ics. 6 and 7. Thus, setting $U_0(x)$ and $\dot{U}_0(x)$ to zero respectively for simplicity and substituting $f_1(t)$, $f_3(t)$, $g_1(x)$ and $g_3(x)$ into the Ics, one obtains $Z(x,0)=0$ and $\frac{\partial}{\partial t}Z(x,0)=-\Omega$ 50

These are the simplified initial conditions for our beam model having simple supports at both ends. Note that, for simplicity, we have set A=B=1 in $f_1(t)$ and $f_3(t)$. Taking the generalized integral transform of the above Ics and using 10 one obtains $\overline{Z}(m,0)=0$, $\overline{Z}_t(m,0)=c^0$ and $c^0=-\frac{\Omega L}{m\pi}\left(1+(-1)^{m+1}\right)$

51

Substituting equations 48, 49 and 51into 40, one obtains

$$\left(1+\frac{r^2m^2\pi^2}{L^2}\right)\overline{Z}_{tt}(m,t)+\left(\frac{m^4\pi^4}{L^4}\frac{EI}{\mu}+\frac{N}{\mu}\frac{m^2\pi^2}{L^2}\right)\overline{Z}(m,t)+\varepsilon\Bigg[\overline{Z}_{tt}(m,t)-\frac{m^2u^2\pi^2}{L^2}\overline{Z}(m,t)$$

$$+\sum_{k=1}^{\infty}\Bigg\{2Sin\frac{m\pi ut}{L}Sin\frac{k\pi ut}{L}\overline{Z}_{tt}(k,t)+\left(\frac{8muk}{(m^2-k^2)}+\sum_{n=1}^{\infty}\frac{16muk\pi^2\left[n^2+k_2-m^2\right]}{\pi^2\left[(n+k)^2-m^2\right]\left[(n+k)^2-m^2\right]}Cos\frac{n\pi ut}{L}\right)\overline{Z}_t(k,t)$$

$$\frac{2k^2u^2\pi^2}{L^2}Sin\frac{m\pi ut}{L}Sin\frac{k\pi ut}{L}\overline{Z}(k,t)\Bigg\}\Bigg]=PSin\frac{m\pi ut}{L}-$$

$$[G_a^*(t)-G_b^*(t)+G_c^*(t)+G_d^*(t)-G_e^*(t)+G_f^*(t)+G_g^*(t)+G_h^*(t)] \qquad 52$$

Where

$$G_a^*(t)=B_1\sum_{i=1}^{1,3}f_i(t)\int_0^L\frac{d^4}{dx^4}g_i(x)Sin\frac{m\pi x}{L}dx,\ G_b^*(t)=B_2\sum_{i=1}^{1,3}f_i(t)\int_0^L\frac{d^2}{dx^2}g_i(x)Sin\frac{m\pi x}{L}dx$$

$$G_c^*(t)=\sum_{i=1}^{1,3}\ddot{f}_i(t)\int_0^L Sin\frac{m\pi x}{L}dx \quad G_d^*(t)=B_3\sum_{i=1}^{1,3}\ddot{f}_i(t)\int_0^L g_i(x)Sin\frac{m\pi x}{L}dx$$

$$G_e^*(t)=r^2\sum_{i=1}^{1,3}\ddot{f}_i(t)\int_0^L\frac{d^2}{dx^2}g_i(x)Sin\frac{m\pi x}{L}dx,\ G_f^*(t)=\frac{M}{\mu}\sum_{i=1}^{1,3}\ddot{f}_i(t)\int_0^L\delta(x-ut)g_i(x)Sin\frac{m\pi x}{L}dx$$

$$G_g^*(t)=\frac{2MU}{\mu}\sum_{i=1}^{1,3}\dot{f}_i(t)\int_0^L\delta(x-ut)\frac{d}{dx}g_i(x)Sin\frac{m\pi x}{L}dx$$

$$G_h^*(t)=\frac{MU^2}{\mu}\sum_{i=1}^{1,3}\dot{f}_i(t)\int_0^L\delta(x-ut)\frac{d^2}{dx^2}g_i(x)Sin\frac{m\pi x}{L}dx \qquad 53$$

In view of 49, equations 53 were evaluated thus:

$$G_a^*(t)=0 \quad ,G_b^*(t)=0,\ G_e^*(t)=0 \quad ,G_c^*(t)=\ddot{f}_1(t)\left[N_1-\frac{1}{L}N_2\right]+\frac{1}{L}\ddot{f}_3(t)N_2$$

$$G_d^*(t)=B_3f_1(t)\left[N_1-\frac{1}{L}N_2\right]+B_3\frac{1}{L}\ddot{f}_3(t)N_2 \qquad 54$$

where

$$N_1=\int_0^l \sin\frac{m\pi x}{L}dx=\frac{L}{m\pi}\left(1+(-1)^{m+1}\right)\text{.....}N_2=\int_0^l x\sin\frac{m\pi x}{L}dx=\frac{L^2}{m\pi}(-1)^{m+1}$$

$$N_3=\int_0^l \cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx=\begin{bmatrix} 0 & \textit{for } (m\pm n)\textit{ even} \\ \dfrac{2mL}{(m^2-n^2)\pi} & \textit{for } (m\pm n)\textit{ odd} \end{bmatrix} \qquad 55$$

Using the series representation of the Dirac Delta function $\delta(x-ut)$, it is straight forward to show that

$$g_1(x)\delta(x-ut)=\frac{1}{L}-\frac{x}{L^2}+\frac{2}{L}\sum_{n=1}^{\infty}Cos\frac{n\pi ut}{L}Cos\frac{n\pi x}{L}-\frac{2x}{L^2}\sum_{n=1}^{\infty}Cos\frac{n\pi ut}{L}Cos\frac{n\pi x}{L} \qquad 56$$

and $g_3(x)\delta(x-ut)=\dfrac{x}{L}\left[\dfrac{1}{L}+\dfrac{2}{L}\displaystyle\sum_{n=1}^{\infty}Cos\frac{n\pi ut}{L}Cos\frac{n\pi x}{L}\right]$ 57

$$G_f^*(t)=\frac{M}{L\mu}\left[\ddot{f}_i(t)\left[N_1+2N_3\sum_{n=1}^{\infty}\cos\frac{n\pi ut}{L}\right]+\frac{1}{L}\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\left[N_2+2N_4\sum_{n=1}^{\infty}\cos\frac{n\pi ut}{L}\right]\right]$$

$$G_g^*(t)=\frac{2MU}{\mu L}\left(\dot{f}_3(t)-\dot{f}_1(t)\right)\sin\frac{m\pi ut}{L} \qquad \text{and} \qquad G_h^*(t)=0 \qquad 58$$

Where

$$\mathrm{N}_4=\int_0^l x\,cos\frac{n\pi x}{L}\,sin\frac{m\pi x}{L}dx=\frac{-L^2}{2\pi}\left[\frac{(m-n)(-1)^{m+n}+(m+n)(-1)^{m-n}}{(m^2-n^2)}\right] \qquad 59$$

Substituting equations 53 to 59 into 52 after some rearrangements, one obtains

$$\left(1+\frac{r^2m^2\pi^2}{L^2}\right)\overline{Z}_{tt}(m,t)+\left(\frac{m^4\pi^4}{L^4}\frac{EI}{\mu}+\frac{N}{\mu}\frac{m^2\pi^2}{L^2}\right)\overline{Z}(m,t)+\varepsilon\left[\overline{Z}_{tt}(m,t)-\frac{m^2u^2\pi^2}{L^2}\overline{Z}(m,t)\right.$$

$$+\sum_{k=1}^{\infty}\left\{2Sin\frac{m\pi ut}{L}Sin\frac{k\pi ut}{L}\overline{Z}_{tt}(k,t)+\frac{2k^2u^2\pi^2}{L^2}Sin\frac{m\pi ut}{L}Sin\frac{k\pi ut}{L}\overline{Z}(k,t)\right.$$

$$\left.\left.\left(\frac{8muk}{(m^2-k^2)}+\sum_{n=1}^{\infty}\frac{16muk\pi^2\left[n^2+k_2-m^2\right]}{\pi^2\left[(n+k)^2-m^2\right]\left[(n+k)^2-m^2\right]}Cos\frac{n\pi ut}{L}\right)\overline{Z}_t(k,t)\right\}\right]$$

$$=\frac{mg}{\mu}Sin\frac{m\pi ut}{L}-\left[\frac{L}{m\pi}\left(\ddot{f}_1(t)+\ddot{f}_3(t)(-1)^{m+1}\right)+\frac{k}{\mu}\frac{L}{m\pi}\left(f_1(t)+f_3(t)(-1)^{m+1}\right)\right]-$$

$$+\frac{ML}{\mu}\left[\frac{L}{m\pi}\left(\overset{+\cdot\cdot}{f}_1(t)+\ddot{f}_3(t)(-1)^{m+1}\right)+4\,\ddot{f}_1(t)\sum_{n=1}^{\infty}\frac{M}{(m^2-n^2)\pi}Cos\frac{n\pi ut}{L}\right.$$

$$\left.+\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\sum_{n=1}^{\infty}L\frac{\left\{(m+n)(-1)^{m-n}-(m-n)(-1)^{m+n})\right\}}{\pi(m^2-n^2)}Cos\frac{n\pi ut}{L}+\frac{2U}{L}\left(\dot{f}_3(t)-\dot{f}_1(t)\right)Sin\frac{m\pi ut}{L}\right] \qquad 60$$

which is the transformed equation governing the problem of simply supported Rayleigh beam traversed by mass M traveling at a uniform velocity u. To solve equation 60 two cases are considered; (a) Moving force case and (b) Moving mass case respectively.

Simply Supported Rayleigh Beam Traversed by Moving Force

This model requires the inertia effect of the moving mass M as negligible. Thus, in equation 60, ε is set to zero. On this consideration, the transformed equation 60 reduces to

$$\left(1+\frac{r^2m^2\pi^2}{L^2}\right)\overline{z}_{tt}(m,t)+\left(\frac{m^4\pi^4}{L^4}\frac{EI}{\mu}-\frac{k}{\mu}+\frac{N}{\mu}\frac{m^2\pi^2}{L^2}\right)\overline{z}(m,t)$$

$$= PSin\frac{m\pi ut}{L} - \frac{L}{m\pi}\left[\left(\ddot{f}_1(t) + \ddot{f}_3(t)(-1)^{m+1}\right) + \frac{k}{\mu}\left(f_1(t) + f_1(t)(-1)^{m+1}\right)\right] \qquad 61$$

Further rearrangement yields

$$\bar{z}_{tt}(m,t) + \frac{\left(\frac{m^4\pi^4}{L^4}\frac{EI}{\mu} - \frac{k}{\mu} + \frac{N}{\mu}\frac{m^2\pi^2}{L^2}\right)}{\left(1 + \frac{r^2m^2\pi^2}{L^2}\right)}\bar{z}(m,t) = \frac{P}{\left(1 + \frac{r^2m^2\pi^2}{L^2}\right)}Sin\frac{m\pi ut}{L} - \frac{L}{m\pi\left(1 + \frac{r^2m^2\pi^2}{L^2}\right)}$$

$$\times\left[\left(\ddot{f}_1(t) + \ddot{f}_3(t)(-1)^{m+1}\right) + \frac{k}{\mu}\left(f_1(t) + f_1(t)(-1)^{m+1}\right)\right] \qquad 62$$

From definition of $f_1(t)$ and $f_3(t)$ in equations 49, it is evident that

$$\begin{aligned} &\dot{f}_1(t) = \Omega Cos\Omega t \ , \qquad \dot{f}_3(t) = e^{-\beta t}(\Omega Cos\Omega t - \beta CSin\Omega t) \\ &\ddot{f}_1(t) = \Omega^2 Sin\Omega t \quad \ddot{f}_3(t) = e^{-\beta t}\left[(\beta^2 - \Omega^2)Sin\Omega t - 2\beta\Omega Cos\Omega t\right] \end{aligned} \qquad 63$$

Substituting equation 63 into equation 62, on simplification and rearrangement, one obtains

$$\bar{z}_{tt}(m,t) + \gamma_{mf}^2\bar{z}(m,t) = P_f Sin\frac{m\pi ut}{L} + C_{f6}Sin\Omega t - C_{f7}e^{-\beta t}Sin\Omega t + C_{f3}e^{-\beta t}Cos\Omega t \qquad 64$$

where

$$\gamma_{mf}^2 = \frac{\left(\frac{m^4\pi^4}{L^4}\frac{EI}{\mu} - \frac{k}{\mu} + \frac{N}{\mu}\frac{m^2\pi^2}{L^2}\right)}{\left(1 + \frac{r^2m^2\pi^2}{L^2}\right)} \ , \qquad 65$$

$$P_f = \frac{P}{\left(1 + \frac{r^2m^2\pi^2}{L^2}\right)} \ , C_{f3} = \frac{2\beta\Omega L}{m\pi\left(1 + \frac{r^2m^2\pi^2}{L^2}\right)} \ C_{f6} = \frac{L}{m\pi\left(1 + \frac{r^2m^2\pi^2}{L^2}\right)}\left(\Omega^2 - \frac{k}{\mu}\right) \ ,$$

$$C_{f7} = \frac{L(-1)^{m=1}}{m\pi\left(1 + \frac{r^2m^2\pi^2}{L^2}\right)}\left((\beta^2 - \Omega^2) + \frac{k}{\mu}\right) \qquad 66$$

To obtain the solution of equation 64, it is subjected to Laplace transformation defined as

$$(\tilde{\cdot}) = \int_0^\infty (\cdot)e^{-st}dt$$

s is the Laplace parameter. Applying the transformed form of initial conditions

$$\bar{Z}(m,0) = 0 = \bar{Z}_t(m,0) \qquad 67$$

one obtains the simple algebraic equation given by

$$\bar{Z}(m,s) = \frac{1}{s^2 + \gamma_{mf}^2}\left[\frac{P_f Z_1}{s^2 + Z_1^2} + \frac{C_{f6}\Omega}{s^2 + \Omega^2} - \frac{C_{f7}\Omega}{(s+\beta)^2 + \Omega^2} + \frac{C_{f3}(s+\beta)}{(s+\beta)^2 + \Omega^2}\right] \qquad 68$$

Where $Z_1 = \frac{m\pi u}{L}$ 69

Thus, our problem reduces to that of finding the Laplace inversion of 68. To do this, we adopt the representation

$$\tilde{f}(s) = \frac{1}{s^2 + \gamma_{mf}^2}, ..and \qquad \tilde{g}(s) = \frac{P_f Z_1}{s^2 + Z_1^2} + \frac{C_{f6}\Omega}{s^2 + \Omega^2} + \frac{C_{f7}\Omega}{(s+\beta)^2 + \Omega^2} + \frac{C_{f3}(s+\beta)}{(s+\beta)^2 + \Omega^2} \qquad 70$$

So that the Laplace inversion of 68 is the convolution of $\tilde{f}(s)$ and $\tilde{g}(s)$. this convolution denoted by $f * g$, is represented by the integral

$$\int_0^t f(t-\tau)g(\tau)d\tau \qquad 71$$

When we introduce above convolution theory, $\overline{Z}(m,t)$ is easily expressed as a sum of eight integrals, namely,

$$\overline{Z}(m,t) = \frac{1}{\gamma_{mf}}\left[X_a - X_b + X_c - X_d - X_e + X_f + X_g - X_h\right] \qquad 72$$

Where $x_{i=a,b,c,d,e,f}$ are integrals which when solved gives

$$X_a = -\frac{p_f Sin Z_1 t}{2}\left[\frac{Cos(\gamma_{mf}+Z_1)t}{(\gamma_{mf}+Z_1)} + \frac{Cos(\gamma_{mf}+Z_1)t}{(\gamma_{mf}-Z_1)} - \frac{2\gamma_{mf}}{(\gamma_{mf}^2-Z_1^2)}\right]$$

$$X_b = \frac{p_f Cos Z_1 t}{2}\left[\frac{Sin(\gamma_{mf}-Z_1)t}{(\gamma_{mf}-Z_1)} \; \frac{Sin(\gamma_{mf}+Z_1)t}{(\gamma_{mf}+Z_1)}\right]$$

$$X_c = -\frac{C_{f6} Sin\Omega t}{2}\left[\frac{Cos(\gamma_{mf}+\Omega)t}{(\gamma_{mf}+\Omega)} + \frac{Cos(\gamma_{mf}-\Omega)t}{(\gamma_{mf}-\Omega)} - \frac{2\gamma_{mf}}{(\gamma_{mf}^2-\Omega^2)}\right]$$

$$X_d = \frac{C_{f6} Cos\Omega t}{2}\left[\frac{Sin(\gamma_{mf}-\Omega)t}{(\gamma_{mf}-\Omega)} - \frac{Sin(\gamma_{mf}+\Omega)t}{(\gamma_{mf}+\Omega)}\right]$$

$$X_e = -\frac{C_{f7}(\gamma_{mf}+\Omega)Sin\gamma_{mf}t}{2\left[(\gamma_{mf}+\Omega)^2+\beta^2\right]}\left[e^{-\beta t}Cos(\gamma_{mf}+\Omega)t + \frac{\beta e^{-\beta t}Sin(\gamma_{mf}+\Omega)t}{(\gamma_{mf}+\Omega)} - 1\right]$$

$$+\frac{C_{f7}(\gamma_{mf}-\Omega)Sin\gamma_{mf}t}{2\left[(\gamma_{mf}-\Omega)^2+\beta^2\right]}\left[e^{-\beta t}Cos(\gamma_{mf}-\Omega)t + \frac{\beta e^{-\beta t}Sin(\gamma_{mf}-\Omega)t}{(\gamma_{mf}-\Omega)} - 1\right]$$

$$X_f = \frac{C_{f7}(\gamma_{mf}-\Omega)Cos\gamma_{mf}t}{2\left[(\gamma_{mf}-\Omega)^2+\beta^2\right]}\left[e^{-\beta t}Sin(\gamma_{mf}-\Omega)t - \frac{\beta e^{-\beta t}Cos(\gamma_{mf}-\Omega)t}{(\gamma_{mf}-\Omega)} + \frac{\beta}{(\gamma_{mf}-\Omega)}\right]$$

$$-\frac{C_{f7}(\gamma_{mf}+\Omega)Sin\gamma_{mf}t}{2\left[(\gamma_{mf}+\Omega)^2+\beta^2\right]}\left[e^{-\beta t}Sin(\gamma_{mf}+\Omega)t - \frac{\beta e^{-\beta t}Cos(\gamma_{mf}+\Omega)t}{(\gamma_{mf}+\Omega)} + \frac{\beta}{(\gamma_{mf}+\Omega)}\right]$$

$$X_g = \frac{C_{f3}(\gamma_{mf}+\Omega)Sin\gamma_{mf}t}{2\left[(\gamma_{mf}+\Omega)^2+\beta^2\right]}\left[e^{-\beta t}Sin(\gamma_{mf}+\Omega)t - \frac{\beta e^{-\beta t}Cos(\gamma_{mf}+\Omega)t}{(\gamma_{mf}+\Omega)} + \frac{\beta}{(\gamma_{mf}+\Omega)}\right]$$

$$-\frac{C_{f3}(\gamma_{mf}-\Omega)Sin\gamma_{mf}t}{2\left[(\gamma_{mf}-\Omega)^2+\beta^2\right]}\left[e^{-\beta t}Sin(\gamma_{mf}-\Omega)t - \frac{\beta e^{-\beta t}Cos(\gamma_{mf}-\Omega)t}{(\gamma_{mf}-\Omega)} + \frac{\beta}{(\gamma_{mf}-\Omega)}\right]$$

$$X_h = -\frac{C_{f3}(\gamma_{mf}+\Omega)Cos\gamma_{mf}t}{2\left[(\gamma_{mf}+\Omega)^2+\beta^2\right]}\left[e^{-\beta t}Cos(\gamma_{mf}+\Omega)t + \frac{\beta e^{-\beta t}Sin(\gamma_{mf}+\Omega)t}{(\gamma_{mf}+\Omega)} - 1\right]$$

$$+\frac{C_{f3}(\gamma_{mf}-\Omega)Cos\gamma_{mf}t}{2\left[(\gamma_{mf}-\Omega)^2+\beta^2\right]}\left[e^{-\beta t}Cos(\gamma_{mf}-\Omega)t + \frac{\beta e^{-\beta t}Sin(\gamma_{mf}-\Omega)t}{(\gamma_{mf}-\Omega)} - 1\right] \qquad 73$$

Substituting equations 73 into 72, after some simplifications, and rearrangements yields

$$\overline{Z}(m,t) = \frac{P_f}{\gamma_{mf}^{2} - \frac{m^2\pi^2u^2}{L^2}}\left\{Sin\frac{m\pi u}{L}t - \frac{m\pi u}{L\gamma_{mf}}\sin\gamma_{mf}t\right\} + \frac{C_{f6}}{\gamma_{mf}^{2} - \Omega^2}\left\{Sin\Omega t - \frac{\Omega}{\gamma mf}Sin\gamma_{mf}t\right\}$$

$$-C_{F7}\frac{(\gamma_{mf}^{2} - \Omega^2 + \beta^2)e^{-\beta t}Sin\Omega t}{(\gamma_{mf}^{2} + \Omega^2 + \beta 2)^2 - 4\gamma_{mf}^{2}\Omega^2} + \frac{C_{F8}(-2\Omega)e^{-\beta t}Cos\Omega t}{\left[(\gamma_{mf}^{2} + \Omega^2 + \beta^2)^2 - 4\gamma_{mf}^{2}\Omega^2\right]}$$

$$+C_{F7}\frac{\Omega(\gamma_{mf}^{2} - \Omega^2 - \beta^2)}{\gamma_{mf}\left[\left(\gamma_{mf}^{2} + \Omega^2 + \beta^2\right)^2 - 4\gamma_{mf}^{2}\Omega^2\right]}Sin\gamma_{mf}t$$

$$+C_{F7}\frac{2\beta\Omega Cos\gamma_{mf}t}{(\gamma_{mf}^{2} + \Omega^2 + \beta^2)^2 - 4\gamma_{mf}^{2}\Omega^2} + \frac{C_{F3}(\gamma_{mf}^{2} - \Omega^2 + \beta^2)e^{-\beta t}Cos\Omega t}{(\gamma_{mf}^{2} + \Omega^2 + P^2)^2 - 4\gamma_{mf}^{2}\Omega^2}$$

$$+\frac{C_{F3}(-2\Omega)e^{-\beta t}Sin\Omega^{t}}{(\gamma_{mf}^{2} + \Omega^2 + \beta^2)^2 - 4\gamma_{mf}^{2}\Omega^2} + \frac{C_{F3}\beta(\gamma_{mf}^{2}\Omega^2 + \beta^2)Sin\gamma_{mf}t}{\gamma_{mf}(\gamma_{mf}^{2} + \Omega^2 + \beta^2)^2 - 4\gamma_{mf}^{2}\Omega^2} \qquad 74$$

$$+\frac{C_{F3}\Omega(\gamma_{mf}^{2} - \Omega^2 - \beta^2)Cos\gamma_{mf}t}{\gamma_{mf}(\gamma_{mf}^{2} + \Omega^2 + \beta^2)^2 - 4\gamma_{mf}^{2}\Omega^2} + \frac{\Omega L}{m\pi\gamma_{mf}}\left(1 + (-1)^{m+1}\right)Sin\gamma_{mf}t$$

which on inversion yields

$$\overline{Z}(X_{,}t) = \frac{2}{L}\sum_{m=1}^{\alpha}[\overline{Z}(m,t)]Sin\frac{m\pi x}{L} \qquad 75$$

but $$U(x,t) = \overline{Z}(x,t) + \sum_{i=1}^{1,3} f_i(t)g_i(t) \qquad 76$$

Thus, $U(x,t) = \overline{Z}(x,t) + (1 - \frac{x}{L})Sin\Omega t + \frac{x}{L}e^{-\beta t}Sin\Omega t$ 77

Equation 77 is the dynamic response of a simply supported Rayleigh beam to moving force when one end of the beam *(x = 0)* is subjected to a sine-wave transient displacement starting from rest while the other end is subjected to a damped sine-wave transient displacement starting from rest.

Simply Supported Rayleigh Beam Traversed by Moving Mass

In this section, the solution to the entire equation 60 is sought when no terms of the coupled DE is neglected. The modification of the asymptotic methods due to Struble's technique often used in treating weakly homogeneous and non-homogeneous, non-linear oscillatory system in view of this 60 is rearranged and by this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the moving mass. An equivalent free system operator defined by the modified frequency then replaces equation 65 the right hand side of equation 60 is set to zero then we considered a parameter $\lambda < 1$ for any arbitrary ratio λ defined as

$\lambda = \frac{\varepsilon}{1+\varepsilon}$, $\varepsilon = \lambda + 0(\lambda^2)$ and the resulting equation was solved using Laplace Transform and convolution theory by using the same argument in the moving force case and from result in Ajibola

(2017) and in literatures comparing the two examples Moving force and Moving Mass respectively hence the discussion of the analytical solution and the Numerical calculations and the results in graphs.

Discussion of the Analytical Solution

When the undamped system like our problem is studied, it is desirable to examine the dynamic response of the dnamical system which grows without bound and referred to as resonance when it occurs. Equation 75 clearly shows that the simply supported elastic Rayleigh beams transverse by a moving force will be in state of resonance wheneer.

$$\gamma_{mf} = \frac{m\pi u}{L} \tag{78}$$

While the same beam under the action of moving mass experiences resonance effect when

$$\alpha_m = \frac{m\pi u}{L} \tag{79}$$

Hence,

$$\alpha_m = \frac{\gamma_{mf}}{2}\left[1 - \frac{1}{1+\frac{r^2m^2\pi^2}{L^2}} + \frac{2\lambda}{\gamma_{mf}^2}\left(\frac{\frac{m^2u^2\pi^2}{L^2} + \gamma_{mf}^2}{\left(1+\frac{r^2m^2\pi^2}{L^2}\right)}\right)\right] \tag{80}$$

is the modified natural frequency representing the frequency of the free system due to the presence of the moving mass. This implies that

$$\gamma_{mf}^2 = \frac{\left(\frac{m^4\pi^4}{L^4}\frac{EI}{\mu} - \frac{k}{\mu} + \frac{N}{\mu}\frac{m^2\pi^2}{L^2}\right)}{\left(1+\frac{r^2m^2\pi^2}{L^2}\right)} \tag{81}$$

It is therefore evident, that for the same natural frequency, the critical speed for the system of a simply supported elastic beam on an elastic foundation and traversed by a moving force is greater than that traversed by moving mass. Thus, resonance is reached earlier in the moving mass system than in the moving force system.

Numerical Calculation and Discussion of the Results

Illustrating the analytical results in dynamics of structures and Engineering designs for this example considered, the uniform Rayleigh beam is length L=12.192m, the load velocity u=8.123 and $E = 2109*10^9 kg/m$.

Axial force N are varied between 0 and 2000000 and foundation moduli K varied between 0 and 400000.The traverse deflections are calculated and plotted against time for values of N and K respectively.

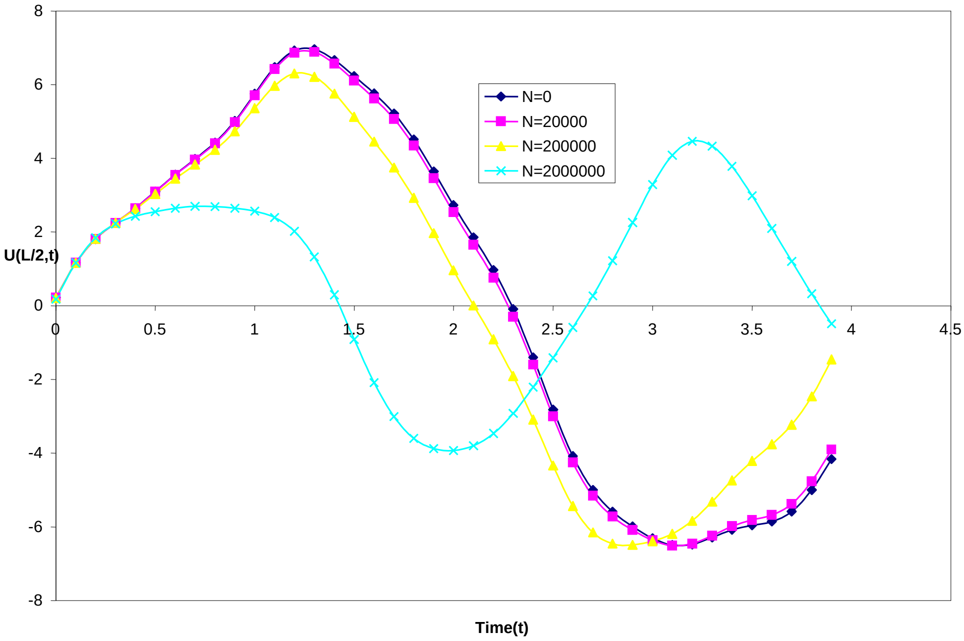

Fig 2.1: transverse displacement response of simply supported moving force of a uniform Rayleigh beam for various values of N and fixed value of K=40000

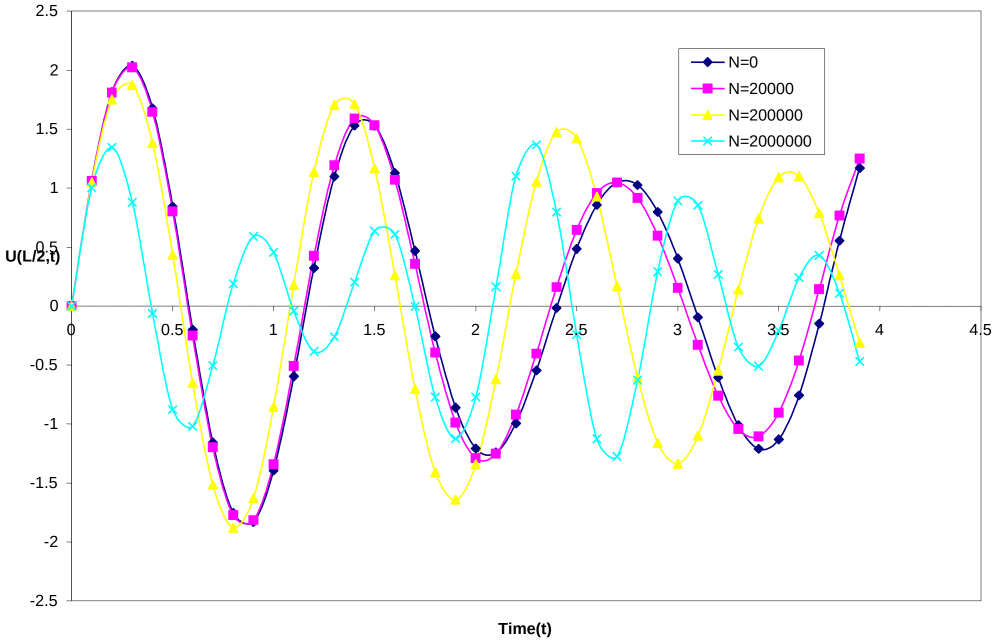

Fig2.2: Transverse displacement response of simply supported moving mass uniform beam for various values of axial force N and for fixed value of foundation moduli K=40000

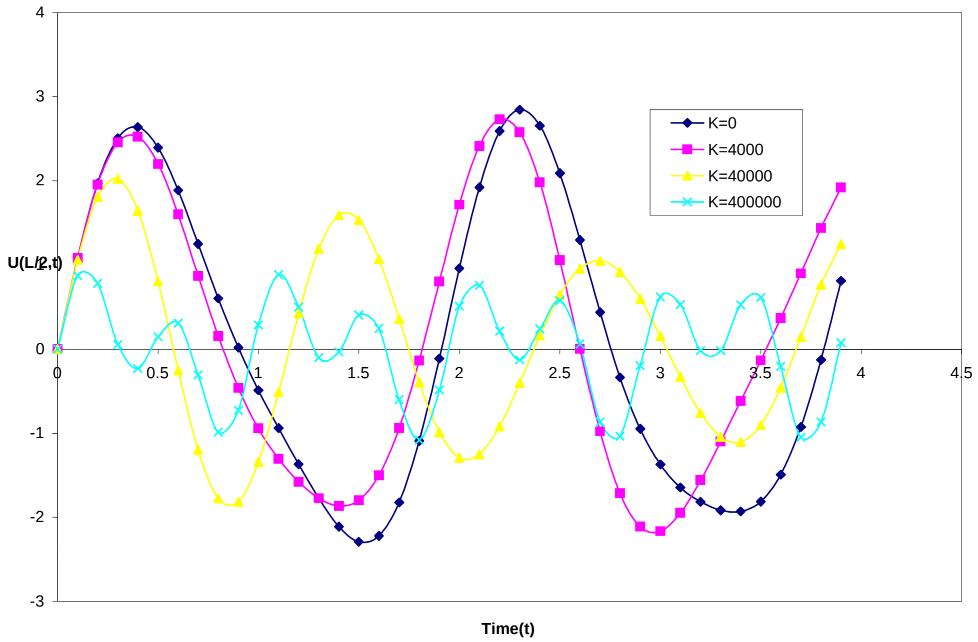

Fig2. 3: Deflection profile of the simply supported moving mass of a uniform Rayleigh beam For various values of K and for fixed value of N=20000

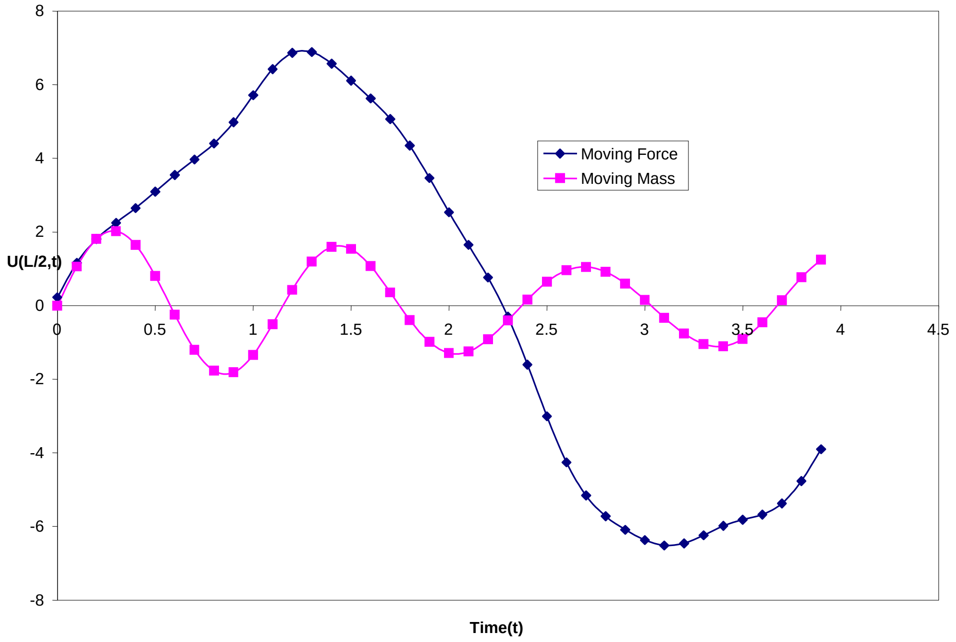

Fig 2.4: Comparing the Moving force and Moving Mass simply supported uniform beam for fixed value of K and N

Fig2.1 , displayed the transverse displacement response of simply supported moving force of a uniform Rayleigh beam for various values of N and fixed value of K =40000. The response amplitude decreases as the values of N increases. Fig 2.2 revels deflection profile of the simply supported moving force for various values of K and for fixed value of N=20000. The graph shows that the response amplitude decreases as the values of K increases. However fig2.3 depicts the transverse displacement response of simply supported moving mass of a uniform Rayleigh beam for various values of N and fixed value of K =40000, from the graph it shows that the response amplitude decreases as the values of N increases. More so, fig 2.4 reflects the deflection profile of the simply supported moving mass for various values of K when N is fixed at N=20000. It is evident from the graph that the response amplitude decreases as the values of K increases. Comparatively, the moving force and moving mass simply supported uniform Rayleigh beams for fixed value of K and N are shown in fig2.5 which reveals that the response amplitudes of moving mass is higher than that of the moving force.

Conclusions

The problem of dynamical analysis of Rayleigh beams with time-dependent boundary conditions when they are under the action of concentrated loads is considered in this article. The governing equations are non-homogeneous fourth order partial differential equations with variable and singular coefficients with non-homogeneous boundary conditions. The main objective is to obtain analytical solutions to the dynamical problems as such solutions often shed light on vital information about the vibrating systems. In particular, a Uniform Rayleigh beam model was considered and the beam was assumed to be reinforced and resting on Uniform elastic foundation moduli. This class of problems is characterized by the fact that the boundary conditions are not stationary and on this account solutions are not in general obtainable by the classical methods of separation of variables. As such, an approach due to Mindlin and Goodman [3] was extended to transform the governing non-homogeneous partial differential equations with non-homogeneous bcs to non-homogeneous partial differential equations with homogeneous bcs.

Furthermore, for this model the solution technique is based firstly, on the generalized integral transformation which was used to remove the singularity in the Governing equation and to reduce it to a sequence of second order differential equation with variable coefficients. This second order differential equation was then simplified using the modification of the Struble's asymptotic technique. The methods of Laplace transformation and the convolution theories were then employed to obtain the analytical solution of our dynamical problem.

Illustrative example of simply supported end conditions which is commonly encountered in engineering practice was presented. Illustrated bcs, solutions for both moving force and moving mass problems were obtained and compared.

Analyses of the approximate analytical solutions obtained were analysed and the resonance amplitudes for the dynamical systems obtained. The influence of the rotator inertia r 3 and foundation moduli k on the dynamic response of the uniform Rayleigh beam having time dependent boundary conditions and under the actions of moving concentrated loads were investigated. The transverse displacements for the moving force and the moving mass models were calculated and presented in plotted curves and the study exhibits the following interesting results:

(a) As the rotatory inertia correction factor R increases, the displacement response reduces for both moving force and moving mass cases.
(b) Illustrative examples considered revealed that the moving force solution is not an upper bound for the accurate solution of the moving mass cases in uniform Rayleigh beams problems. Hence, the non-reliability of moving force solution as a safe approximation to the moving mass models is confirmed.
(c) The response amplitudes of a uniform Rayleigh beam having time dependent bcs and resting on elastic foundation decreases as K increases for both variants of time-dependent boundary conditions.
(d) For fixed rotator y inertia correction factor and foundation moduli, the transverse displacements of the uniform Rayleigh beam under the action of moving concentrated forces and masses decrease as the N increases.
(e) The two illustrative examples considered, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem. Hence resonance is reached earlier in moving mass problem, in this regard there is need to consider the inertia effect of the moving load.

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