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On the Behavioral Analysis of Saturation Term on Mathematical Model

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Abstract
The paper analyses the behavior of a mathematical epidemic model. The particular interest is the criteria for obtaining the basic reproduction number of the modified model with the local and global stabilities of the model. The numerical results show the effect of sociological and physiological parameters on the model by using Runge kutta of order 4, and that saturation term which analyses the behavior, has appreciable effects on the mathematical model.

Keywords: Basic Reproduction Number; Disease Free Equilibrium; Saturation Terms; Lyapunov Functions.

Introduction

Mojeeb *et al.* (2017) studied simple mathematical model for malaria transmission with a bilinear incidence rate. Kolawole and Olayiwola (2016) investigated the numerical simulation of Saturation term on SEIRS epidemic model with the bahavioural analysis and it was concluded that saturation term plays vital role in SEIRS epidemic model. Nita and Guptta (2013) investigated SEIR model and simulation for vector borne diseases. (Adebimpe *et al.*, 2013 and Sarah, 2012) studied SEIR epidemic model with saturated incidence rate and limited resources for treatment. They concluded that, saturation terms which are the parameter that describe the physiological and sociological effects play vital role in disease eradication. De la sean *et al.* (2012) studied and analyzed an SEIR epidemic model with a general feedback vaccination law. They concluded that vaccination plays vital role in the epidemic model.

The Basic Mathematical Model

In this paper, a model was adopted and modified by incorporating an incidence rate which include saturation term m to investigate its behavior on the model.

Existing model of Mojeeb (2017)

$$\frac{dS}{dt} = \wedge - \beta SI - \mu S$$
$$\frac{dE}{dt} = \beta SI - (\alpha_1 + \mu)E$$
$$\frac{dI}{dt} = \alpha_1 E - (\alpha_2 + \mu + \delta)I$$
$$\frac{dR}{dt} = \alpha_2 I - \mu R \qquad 1$$

Proposed Model

$$\frac{dS}{dt} = \wedge - \frac{\beta SI}{1+mS} - \mu S$$
$$\frac{dE}{dt} = \frac{\beta SI}{1+mS} - (\alpha_1 + \mu)E$$
$$\frac{dI}{dt} = \alpha_1 E - (\alpha_2 + \mu + \delta)I$$
$$\frac{dR}{dt} = \alpha_2 I - \mu R \qquad 2$$

Disease Free Equilibrium (DFE)

At disease free equilibrium, $I = 0, E = 0$,

From equation (2), we have;

$$\wedge - \frac{\beta SI}{1+mS} - \mu S = 0$$

$$\wedge - \mu S = 0$$

$$S = \frac{\wedge}{\mu} \qquad 3$$

Hence the disease-free equilibrium gives,

$$E_0 = (S, E, I) = (\frac{\wedge}{\mu}, 0, 0) \qquad 4$$

Endemic Equilibrium

To obtain an endemic equilibrium, $I \neq 0$

From equation (2), to get E^*

$$\alpha_1 E - (\alpha_2 + \mu + \delta) I = 0$$

$$\alpha_1 E = (\alpha_2 + \mu + \delta) I$$

$$E^* = \frac{(\alpha_2 + \mu + \delta) I^*}{\alpha_1}$$

$$E^* = \frac{(\alpha_2 + \mu + \delta)}{\alpha_1} [\frac{\wedge \alpha_1 \beta - (\wedge m + \mu)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 - m(\alpha_1 + \mu)^2 (\alpha_2 + \mu + \delta)^2}] \qquad 5$$

From equation (2), to get S^*

$$\frac{\beta SI}{1+mS} - (\alpha_1 + \mu) E = 0$$

$$\frac{\beta SI}{1+mS} = (\alpha_1 + \mu) E$$

$$\beta SI = (1+mS)(\alpha_1 + \mu) E$$

$$S^* = \frac{(\alpha_1 + \mu) E^*}{\beta I^* - m(\alpha_1 + \mu) E^*}$$

$$S^* = \frac{(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 \beta - m(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)} \qquad 6$$

From equation (2) to get I^*,

$$\wedge - \frac{\beta SI}{1+mS} - \mu S = 0$$

$$\wedge - \frac{\beta SI}{1+mS} - \mu S$$

$$\wedge - \mu S = \frac{\beta SI}{1+mS}$$

$$I^{*} = \alpha_1[\frac{\wedge \alpha_1 \beta - (\wedge m + \mu)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 - m(\alpha_1 + \mu)^2 (\alpha_2 + \mu + \delta)^2}] \qquad 7$$

$$(S^{*}, E^{*}, I^{*}) = (\frac{(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 \beta - m(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}, \frac{(\alpha_2 + \mu + \delta)}{\alpha_1}[\frac{\wedge \alpha_1 \beta - (\wedge m + \mu)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 - m(\alpha_1 + \mu)^2 (\alpha_2 + \mu + \delta)^2}],$$

$$\alpha_1[\frac{\wedge \alpha_1 \beta - (\wedge m + \mu)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 - m(\alpha_1 + \mu)^2 (\alpha_2 + \mu + \delta)^2}]) \qquad 8$$

Basic Reproduction Number R_0

It is derived using next generation matrix method, that is;

$$R_0 = F.V^{-1} \quad . \qquad 9$$

$$F = \begin{bmatrix} 0 & \frac{\beta \wedge}{\mu + m \wedge} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} (\alpha_1 + \mu) & 0 \\ \alpha_1 & -(\alpha_2 + \mu + \delta) \end{bmatrix} \qquad 10$$

The inverse of V is obtained as

$$V^{-1} = \frac{-1}{(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)} \begin{bmatrix} (\alpha_2 + \mu + \delta) & -\alpha_1 \\ 0 & (\alpha_1 + \mu) \end{bmatrix} \qquad 11$$

Hence,

$$R_0 = FXV^{-1} = \begin{bmatrix} 0 & \frac{\beta \wedge}{\mu + m \wedge} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{-1}{(\alpha_1 + \mu)} & 0 \\ \frac{\alpha_1}{(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)} & \frac{-1}{(\alpha_2 + \mu + \delta)} \end{bmatrix} \qquad 12$$

$$R_0 = \begin{bmatrix} \frac{\beta \wedge \alpha_1}{(\mu + m\wedge)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)} & \frac{\beta \wedge}{(\mu + m\wedge)(\alpha_2 + \mu + \delta)} \\ 0 & 0 \end{bmatrix}$$

The equation with the dominant eigen value becomes,

$$R_0 = \frac{\beta \wedge \alpha_1}{(\mu + m\wedge)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)} \qquad 13$$

Local Stability of Disease-Free Equilibrium

The system of equation (2) was linearized by setting

$$S - S_1 = x, E = E, I = I,$$

$$S = x + S_1 \Rightarrow \frac{d}{dt}(x + S_1) = \frac{dx}{dt} \qquad 14$$

$$\frac{dx}{dt} = -\beta I(x + S_1)(1 + m(x + S_1))^{-1} - \mu(x + S_1)$$

$$\frac{dE}{dt} = \beta I(x + S_1)(1 + m(x + S_1))^{-1} - (\alpha_1 + \mu)E$$

$$\frac{dI}{dt} = \alpha_1 E - (\alpha_2 + \mu + \delta)I \qquad 15$$

Therefore,

$$\frac{dx}{dt} = -\mu x - \beta I S_1 + \text{Non Linear terms}$$

$$\frac{dE}{dt} = -(\alpha_1 + \mu)E + \beta I S_1 + \text{Non Linear terms}$$

$$\frac{dI}{dt} = \alpha_1 E - (\alpha_2 + \mu + \delta)I \qquad 16$$

The resulting Jacobian matrix becomes

$$J(S,E,I) = \begin{bmatrix} -\mu & 0 & -\beta S_1 \\ 0 & -(\alpha_1 + \mu) & \beta S_1 \\ 0 & \alpha_1 & -(\alpha_2 + \mu + \delta) \end{bmatrix} \qquad 17$$

The determinant $\left| J - I\lambda \right| = 0$

$$\begin{vmatrix} -\mu - \lambda & 0 & -\frac{\beta \wedge}{\mu} \\ 0 & -(\alpha_1 + \mu) - \lambda & \frac{\beta \wedge}{\mu} \\ 0 & \alpha_1 & -(\alpha_2 + \mu + \delta) - \lambda \end{vmatrix} = 0 \qquad 18$$

It follows that the characteristic equation becomes

$$\lambda^3 + (3\mu + \alpha_1 + \alpha_2 + \delta)\lambda^2 + (\mu^2(\alpha_2 + \alpha_1 + 2\mu + \delta) + [(\mu - R_0(\mu + m\wedge))(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)]\lambda +$$
$$(\mu - R_0(\mu + m\wedge))(\alpha_1 + \mu)(\alpha_2 + \mu + \delta) = 0$$

We can write the characteristics equation above as:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \qquad 19$$

Where,

$$a_1 = 3\mu + \alpha_1 + \alpha_2 + \delta$$
$$a_2 = (\mu^2(\alpha_2 + \alpha_1 + 2\mu + \delta) + [(\mu - R_0(\mu + m\wedge))(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)]$$
$$a_3 = (\mu - R_0(\mu + m\wedge))(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)$$

Using the Routh-Hurwitz criterion, it can be seen that all the eigen values of the characteristics equation above have negative real part if and only if:

$$a_1 > 0, a_2 > 0, a_3 > 0, a_1 a_2 - a_3 > 0 \qquad 20$$

Therefore, it is locally asymptotically stable if and only if inequalities above are satisfied.

Global Stability of Disease-Free Equilibrium

We consider the Lyapunov function,

$$L = \alpha_1 E + (\alpha_1 + \mu) I$$

$$L^1 = \alpha_1 \frac{dE}{dt} + (\alpha_1 + \mu)\frac{dI}{dt}$$

$$\le (\alpha_1 + \mu)(\alpha_2 + \mu + \delta)[\frac{\beta \wedge \alpha_1}{(\mu + m\wedge)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)} - 1]I$$

$$= (\alpha_1 + \mu)(\alpha_2 + \mu + \delta)[R_0 - 1]I \qquad 21$$

$if \quad R_0 \le 1$

$\text{Ł}^1 \le 0$

Hence the disease-free equilibrium is globally asymptotically stable.

Existence of Endemic Equilibrium

By Linearizing (2), we have

Let $S - S^* = w, E - E^* = x, I - I^* = y, R - R^* = z$

$$\frac{ds}{dt} = \frac{dw}{dt}, \frac{dE}{dt} = \frac{dx}{dt}, \frac{dI}{dt} = \frac{dy}{dt}$$

$$\frac{dw}{dt} = -\frac{\beta(\omega + S^*)(y + I^*)}{1 + m(\omega + S^*)} - \mu(\omega + S^*)$$

$$\frac{dx}{dt} = \frac{\beta(\omega + S^*)(y + I^*)}{1 + m(\omega + S^*)} - (\alpha_1 + \mu)(x + E^*)$$

$$\frac{dy}{dt} = \alpha_1(x + E^*) - (\alpha_2 + \mu + \delta)(y + I^*)$$

The linearized equations are as follows;

$$\frac{dw}{dt} = (-\beta I^* - \mu)w - \beta S^* y + \text{Non Linear terms}$$

$$\frac{dx}{dt} = \beta S^* y - \alpha_1 x - \mu x + \beta I^* w + \text{Non Linear terms}$$

$$\frac{dy}{dt} = \alpha_1 x - y(\alpha_2 + \mu + \delta) \qquad 22$$

From the linearized system, we have

$$J(E^*) = \begin{bmatrix} (-\beta I^* - \mu) & 0 & -\beta S^* \\ \beta I^* & -(\alpha_1 + \mu) & -\beta S^* \\ 0 & \alpha_1 & -(\alpha_2 + \mu + \delta) \end{bmatrix} \qquad 23$$

Where $S^* = \frac{(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 \beta - m(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}, I^* = \alpha_1[\frac{\wedge \alpha_1 \beta - (\wedge m + \mu)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 - m(\alpha_1 + \mu)^2(\alpha_2 + \mu + \delta)^2}])$ 24

Therefore, we have

$$A = \left(\beta\alpha_1 [\frac{\wedge \alpha_1\beta - (\wedge m + \mu)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 - m(\alpha_1 + \mu)^2 (\alpha_2 + \mu + \delta)^2} + \alpha_1 + \mu + \delta \right)$$

$$B = \left(\begin{array}{l} -(\alpha_2 + \delta + \mu)(\beta\alpha_1 (\frac{\wedge \alpha_1\beta - (\wedge m + \mu)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 - m(\alpha_1 + \mu)^2 (\alpha_2 + \mu + \delta)^2}) + \alpha_1) - (\beta{\alpha_1}^2 + \beta\alpha_1\mu) \\ (\frac{\wedge \alpha_1\beta - (\wedge m + \mu)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 - m(\alpha_1 + \mu)^2 (\alpha_2 + \mu + \delta)^2} \end{array} \right)$$

$$C = \left(\alpha_2 + \delta + \mu)(\beta\alpha_1 + \mu)(\frac{\wedge \alpha_1\beta - (\wedge m + \mu)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 - m(\alpha_1 + \mu)^2 (\alpha_2 + \mu + \delta)^2}) + \mu\alpha_1 + \mu^2 \right) +$$

$$\beta\alpha_1 \left(\frac{(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1\beta - m(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)} \right) \left((1 - \beta\alpha_1)(\frac{\wedge \alpha_1\beta - (\wedge m + \mu)(\alpha_1 + \mu)(\alpha_2 + \mu + \delta)}{\alpha_1 - m(\alpha_1 + \mu)^2 (\alpha_2 + \mu + \delta)^2}) \right)$$

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \qquad 25$$

Using the Routh-Hurwitz criterion, it can be seen that all the eigen values of the characteristics equation above have negative real part if and only if:

$$A > 0, B > 0, C > 0, AB - C > 0 \qquad 26$$

Hence, it is locally asymptotically stable if and only if inequalities above are satisfied

Results and Discussion

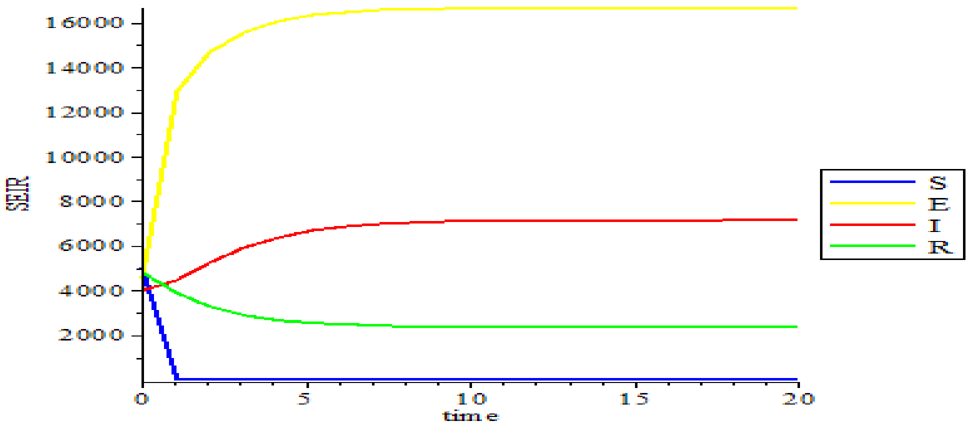

fig 1:Graph of Susceptible(S), Infected(I), Exposed(E), Recovered(R) classagain time(t) with $\beta=0.1$, $\Lambda=10000$, $\mu=0.3$, $\delta=0.3$, $m=0.01$, $\alpha_1 := 0.3$, $\alpha_2 = 0.1$

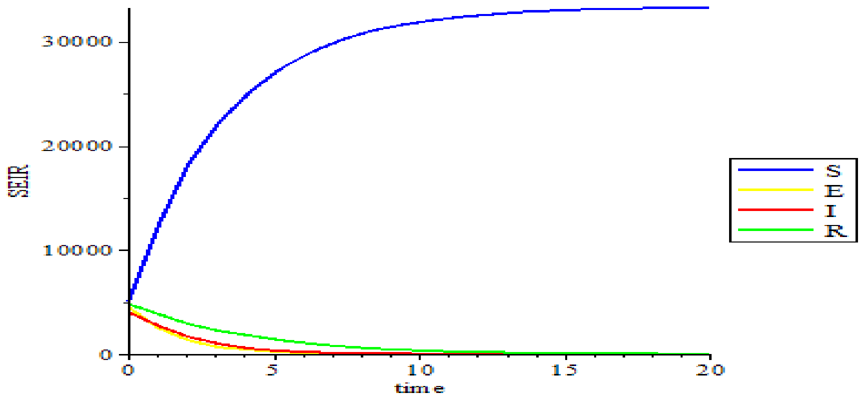

fig 2:Graph of Susceptible(S), Infected(I), Exposed(E), Recovered(R) classagain time(t) with $\beta=0.1$, $\Lambda=10000$, $\mu=0.3$, $\delta=0.3$, $m=10$, $\alpha_1 = 0.3$, $\alpha_2 = 0.1$

Conclusions

From figures 1-2, the simulation result of the saturation terms shows that the lower the saturation term the lower the susceptible compartment. Also, the higher the saturation terms the higher the susceptible compartment. Therefore, saturation terms play a vital role in disease eradication.

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