New Results on Hua's Generalization of Opial-type Inequalities

Rauf Kamilu[1], Kareem Habeeb[2] and Wahab Rafiu Adesola[3]
Department of Mathematics, University of Ilorin, Nigeria
[1]krauf@unilorin.edu.ng

Abstract
In this paper, we establish Hua's generalization of Opial's inequality with the best possible constant and extend the generalization through some known mathematical analysis methods.

Keywords: Opial's Inequality; Best Possible; Hua's Inequality; Boundedness.

Introduction

Inequalities involving integral of functions and their derivatives are the concern of Opial-type inequalities. In 1960, Opial established the following interesting integral inequality:

Let $x \in C^1[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then the following inequality holds:

$$\int_0^h |x(t)x'(t)|\,dt \le \frac{h}{4}\int_0^h (x'(t))^2\,dt. \tag{1}$$

The constant $\frac{h}{4}$ is the best possible.

Since the discovery of the inequality, a lots of efforts have been done on new proofs and extensions in the literature. Researchers such as: Nuriye *et al.* (2012), Saker (2013), Arif *et al.* (2015) and others have worked on generalization of the inequality.

In further simplifying the proof of the inequality which had already been simplified by Olech (1960), Beesack (1962) proved that if x is real and absolutely continuous on $(0,b)$ with $x(a) = x(b) = 0$, then

$$\int_a^b |x(t)||x'(t)|\,dt \le \frac{b}{2}\int_a^b |x'(t)|^2\,dt. \tag{2}$$

Opial's inequality and it's generalization, extension and discretization play an important role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equation.

Inequalities which involves integrals of a function and its derivatives are of great importance in Mathematics with applications in probability. The following are results obtained by Hua (1965):

Theorem 1: If $x(t)$ is absolutely continuous on $[0,h]$ with $x(0) = 0$, then

$$\int_0^h |x(t)x'(t)|\,dt \le \frac{h}{2}\int_0^h (x'(t))^2\,dt \tag{3}$$

Theorem 2: Let $x(t)$ be absolutely continuous on $[0,a]$, and $x(0) = 0$. For $\ell, m \ge 1$, then the following inequality holds:

$$\int_0^a |x(t)|^\ell |x'(t)|^m\,dt \le \frac{m}{\ell+m}a^\ell \int_0^a |x'(t)|^{\ell+m}\,dt \tag{4}$$

Theorem 3: Let $x(t)$ be absolutely continuous on $[0,a]$, and $x(0) = 0$. Further, let ℓ be a positive integer. Then the following inequality holds:

$$\int_0^a |x^\ell(t)x'(t)|\,dt \le \frac{a^\ell}{\ell+1}\int_0^a |x'(t)|^{\ell+1}\,dt \tag{5}$$

Results

We present the following theorems as our main results:

Theorem 4: Let $x(t)$ be absolutely continuous on $[0,a]$, and $x(0)=0$. Then, for $\ell, m \geq 1$, the following inequality holds:

$$\int_0^a |x(t)|^{\ell} |x'(t)|\, dt \leq \frac{a^{\ell}}{\ell+1} \int_0^a |x'(t)|^{\ell+1}\, dt. \tag{6}$$

Proof:

Assume $\ell = 1$, then we have

$$\int_0^a |x(t)x'(t)|\, dt \leq \frac{a}{2} \int_0^a |x'(t)|^2\, dt. \tag{7}$$

Clearly, (7) is Hua's inequality.

Suppose (6) is true for ℓ, then check for $\ell+1$

$$\int_0^a |x^{\ell+1}(t)x'(t)|\, dt = \int_0^a |x^{\ell}(t)x(t)x'(t)|\, dt$$

$$= \int_0^a |x^{\ell}(t)x'(t)x(t)|\, dt \qquad \leq \frac{a^{\ell}}{\ell+1} \int_0^a |x'(t)|^{\ell+1} |x(t)|\, dt$$

$$= \frac{a^{\ell}}{\ell+1} \int_0^a |x(t)|(x'(t))^{\ell+1}|\, dt \qquad \leq \frac{a^{\ell}}{\ell+1} \cdot \frac{a}{2} \int_0^a |x'(t)|^{\ell+1+1}\, dt$$

$$= \frac{a^{\ell+1}}{2\ell+2} \int_0^a |x'(t)|^{(\ell+1)+1}\, dt \qquad \leq \frac{a^{\ell+1}}{\ell+1} \int_0^a |x'(t)|^{(\ell+1)+1}\, dt$$

Since it is true for $\ell+1$, hence $\int_0^a |x(t)|^{\ell} |x'(t)|\, dt \leq \frac{a^{\ell}}{\ell+1} \int_0^a |x'(t)|^{\ell+1}\, dt$ is true for all $\ell \in \mathbb{N}$.

Theorem 5: Let $x(t)$ be absolutely continuous on $[0,a]$, and $x(0)=0$. Then, for $\ell, m \geq 1$, the following inequality holds:

$$\int_0^a |x(t)|^{\ell} |x'(t)|^m\, dt \leq \frac{m}{\ell+m} a^{\ell} \int_0^a |x'(t)|^{\ell+m}\, dt \tag{8}$$

Proof:

Assume $m, \ell = 1$, then we have

$$\int_0^a |x(t)||x'(t)|\, dt \leq \frac{a}{2} \int_0^a |x'(t)|^2\, dt. \tag{9}$$

Clearly, (9) is Hua's inequality.

Fix ℓ and suppose it is true for m, then check for $m+1$

$$\int_0^a |x^{\ell}(t)||x'(t)|^{m+1}\, dt = \int_0^a |x^{\ell}||x'(t)|^m |x'(t)|\, dt \leq \frac{m}{\ell+m} a^{\ell} \int_0^a |x(t)|^{\ell+m} |x'(t)|\, dt$$

$$= \frac{m}{\ell+m} a^{\ell} \int_0^a |x'(t)|^{\ell+m+1}\, dt \ \leq \frac{m+1}{\ell+(m+1)} a^{\ell} \int_0^a |x'(t)|^{\ell+(m+1)}\, dt \tag{10}$$

Since (10) it is true for $m+1$, hence

$$\int_0^a |x(t)|^{\ell} |x'(t)|^m\, dt \leq \frac{m}{\ell+m} a^{\ell} \int_0^a |x'(t)|^{\ell+m}\, dt$$

is true for all $m \in \mathbb{N}$.

Theorem 6: Let $q(t)$ be positive, bounded and non-increasing on [α, τ]. Further, let $x(t)$ be absolutely continuous on [α, τ], and $x(\alpha) = 0$. Then, for $k > 0, n \geq 1$, the following inequality holds:

$$\int_{\alpha}^{\tau} q(t) \,|\, x(t) \,|^{\frac{1}{k}} |\, x'(t) \,|^{n} \, dt \leq \frac{kn}{kn+1} (\tau - \alpha)^{\frac{1}{k}} \int_{\alpha}^{\tau} q(t) \,|\, x'(t) \,|^{\frac{kn+1}{k}} \, dt. \tag{11}$$

Proof:

Let

$$y(t) = \int_{\alpha}^{t} (q(s))^{\frac{kn}{kn+1}} \,|\, x'(s) \,|^{n} \, ds, t \in [\alpha, \tau]. \tag{12}$$

By integrating the right hand side of (12), we have

$$y(t) = (q(t))^{\frac{kn}{kn+1}} \,|\, x(t) \,|^{n} \,. \tag{13}$$

On differentiating (13), we obtain

$$y'(t) = (q(t))^{\frac{kn}{kn+1}} \,|\, x'(t) \,|^{n} \,.$$

Thus, from Hölder's inequality with indices n and $\frac{n}{n-1}$, we have

$$\begin{aligned}
|\, x(t) \,| &\leq \int_{\alpha}^{t} |\, x'(s) \,| \, ds = \int_{\alpha}^{t} (q(s))^{\frac{-k}{k(kn+1)}} (q(s))^{\frac{k}{kn+1}} \,|\, x'(s) \,| \, ds \\
&\leq \left(\int_{\alpha}^{t} (q(s))^{\frac{-k}{kn+1} \cdot \frac{n}{n-1}} ds \right)^{\frac{n-1}{n}} \left(\int_{\alpha}^{t} (q(s))^{\frac{k}{kn+1} \cdot n} \,|\, x'(s) \,|^{n} \, ds \right)^{\frac{1}{n}} \\
&= (q(t))^{\frac{-k}{kn+1}} (\tau - \alpha)^{\frac{n-1}{n}} (y(t))^{\frac{1}{n}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\alpha}^{\tau} q(t) \,|\, x(t) \,|^{\frac{1}{k}} |\, x'(t) \,|^{n} \, dt &= \int_{\alpha}^{\tau} q(t) (q(t))^{\frac{-1}{kn+1}} (\tau - \alpha)^{\frac{n-1}{kn}} (y(t))^{\frac{1}{kn}} y'(t) (q(t))^{\frac{-kn}{kn+1}} dt \\
&\leq (\tau - \alpha)^{\frac{n-1}{kn}} \int_{\alpha}^{\tau} (y(t))^{\frac{1}{kn}} y'(t) dt \\
&= \frac{kn}{kn+1} (\tau - \alpha)^{\frac{n-1}{kn}} (y(\tau))^{\frac{kn+1}{kn}} .
\end{aligned} \tag{14}$$

On the other hand, from Hölder's inequality with indices $kn+1$ and $\frac{kn+1}{kn}$, we have

$$\begin{aligned}
&\int_{\alpha}^{\tau} (q(t))^{\frac{kn}{kn+1}} \,|\, x'(t) \,|^{n} \, dt = \int_{\alpha}^{\tau} (q(t))^{\frac{kn}{kn+1}} (q(t))^{\frac{-kn}{kn+1}} (q(t))^{\frac{kn}{kn+1}} \,|\, x'(t) \,|^{n} \, dt \\
&\leq \left(\int_{\alpha}^{\tau} (q(t))^{\frac{kn}{kn+1} \cdot kn+1} (q(t))^{\frac{-kn}{kn+1} \cdot kn+1} dt \right)^{\frac{1}{kn+1}} \left(\int_{\alpha}^{\tau} (q(t))^{\frac{kn}{kn+1} \cdot \frac{kn+1}{kn}} \,|\, x'(t) \,|^{n \cdot \frac{kn+1}{kn}} \, dt \right)^{\frac{kn}{kn+1}} \\
&= \left(\int_{\alpha}^{\tau} dt \right)^{\frac{1}{kn+1}} \left(\int_{\alpha}^{\tau} q(t) \,|\, x'(t) \,|^{\frac{kn+1}{k}} \, dt \right)^{\frac{kn}{kn+1}}
\end{aligned}$$

$$\int_{\alpha}^{\tau} (q(t))^{\frac{kn}{kn+1}} \,|\, x'(t) \,|^{n} \, dt \leq (\tau - \alpha)^{\frac{1}{kn+1}} \left(\int_{\alpha}^{\tau} q(t) \,|\, x'(t) \,|^{\frac{kn+1}{k}} \, dt \right)^{\frac{kn}{kn+1}} \tag{15}$$

Taking $\frac{kn+1}{kn}$ power of both sides in (15) we have

$$\left(\int_{\alpha}^{\tau}(q(t))^{\frac{kn}{kn+1}}|x'(t)|^{n}\,dt\right)^{\frac{kn+1}{kn}} \le (\tau-\alpha)^{\frac{1}{kn}}\int_{\alpha}^{\tau}q(t)|x'(t)|^{\frac{kn+1}{k}}\,dt. \tag{16}$$

Since $y(t)=(q(t))^{\frac{kn}{kn+1}}|x'(t)|^{n}$ in (13). Then, by integrating both sides with respect to t over $[\alpha,\tau]$. Hence, (16) becomes

$$\left(y(\tau)\right)^{\frac{kn+1}{kn}} \le (\tau-\alpha)^{\frac{1}{kn}}\int_{\alpha}^{\tau}q(t)|x'(t)|^{\frac{kn+1}{kn}}\,dt. \tag{17}$$

Combining (14) and (17), we obtain

$$\int_{\alpha}^{\tau}q(t)|x(t)|^{\frac{1}{k}}|x'(t)|^{n}\,dt \le \frac{k}{kn+1}(\tau-\alpha)^{\frac{1}{k}}\int_{\alpha}^{\tau}q(t)|x'(t)|^{\frac{kn+1}{k}}\,dt. \tag{18}$$

Remark 1: For $\alpha=0,\tau=a$ and $k, q(t)=1$, Theorem 6 reduces to Theorem 1.

Theorem 7: Let $q(t)$ be positive, bounded and non-increasing on [α,τ]. Further, let $x(t)$ be absolutely continuous on [α,τ], and $x(\alpha)=0$. Then, for $\ell,k>0$, and $m\ge 1$ the following inequality holds:

$$\int_{\alpha}^{\tau}q(t)|x(t)|^{\frac{\ell}{k}}|x'(t)|^{m}\,dt \le \frac{km}{km+\ell}(\tau-\alpha)^{\frac{\ell}{k}}\int_{\alpha}^{\tau}q(t)|x'(t)|^{\frac{km+\ell}{k}}\,dt. \tag{19}$$

Proof:

Let

$$y(t)=\int_{\alpha}^{t}(q(s))^{\frac{km}{\ell+km}}|x'(s)|^{m}\,ds, t\in[\alpha,\tau]. \tag{20}$$

By integrating the right hand side of (20), we obtain

$$y(t)=(q(t))^{\frac{km}{\ell+km}}|x(t)|^{m}\ . \tag{21}$$

differentiating (21) to obtain

$$y'(t)=(q(t))^{\frac{km}{\ell+km}}|x'(t)|^{m}$$

Thus, from Hölder's inequality with indices m and $\frac{m}{m-1}$, we have

$$|x(t)|\le\int_{\alpha}^{t}|x'(s)|\,ds=\int_{\alpha}^{t}(q(s))^{\frac{-k}{\ell+km}}(q(s))^{\frac{k}{\ell+km}}|x'(s)|\,ds$$

$$\le\left(\int_{\alpha}^{t}(q(s))^{\frac{-k}{\ell+km}\cdot\frac{m}{m-1}}ds\right)^{\frac{m-1}{m}}\left(\int_{\alpha}^{t}(q(s))^{\frac{k}{\ell+km}}|x'(s)|^{m}\,ds\right)^{\frac{1}{m}}$$

$$=(q(t))^{\frac{-k}{\ell+km}}(\tau-\alpha)^{\frac{m-1}{m}}(y(t))^{\frac{1}{m}}$$

Hence,

$$\int_{\alpha}^{\tau}q(t)|x(t)|^{\frac{\ell}{k}}|x'(t)|^{m}\,dt=\int_{\alpha}^{\tau}q(t)(q(t))^{\frac{-\ell}{\ell+km}}(\tau-\alpha)^{\frac{\ell(m-1)}{km}}(y(t))^{\frac{\ell}{km}}y'(t)(q(t))^{\frac{-km}{\ell+km}}dt$$

$$\le(\tau-\alpha)^{\frac{\ell(m-1)}{km}}\int_{\alpha}^{\tau}(y(t))^{\frac{\ell}{km}}y'(t)dt.$$

$$\int_{\alpha}^{\tau} q(t)\,|\,x(t)\,|^{\frac{\ell}{k}}\,|\,x'(t)\,|^{m}\,dt \le \frac{km}{\ell+km}(\tau-\alpha)^{\frac{\ell(m-1)}{km}}\,(y(\tau))^{\frac{\ell+km}{km}} \tag{22}$$

On the other hand, from Hölder's inequality with indices $\dfrac{\ell+km}{\ell}$ and $\dfrac{\ell+km}{km}$, we have

$$\int_{\alpha}^{\tau}(q(t))^{\frac{km}{\ell+km}}\,|\,x'(t)\,|^{m}\,dt = \int_{\alpha}^{\tau}(q(t))^{\frac{km}{\ell+km}}(q(t))^{\frac{-km}{\ell+km}}(q(t))^{\frac{km}{\ell+km}}\,|\,x'(t)\,|^{m}\,dt$$

$$\le \left(\int_{\alpha}^{\tau}(q(t))^{\frac{km}{\ell+km}\cdot\frac{\ell+km}{\ell}}(q(t))^{\frac{-km}{\ell+km}\cdot\frac{\ell+km}{\ell}}\,dt\right)^{\frac{\ell}{\ell+km}}\left(\int_{\alpha}^{\tau}(q(t))^{\frac{km}{\ell+km}\cdot\frac{\ell+km}{km}}\,|\,x'(t)\,|^{m\cdot\frac{\ell+km}{km}}\,dt\right)^{\frac{km}{\ell+km}}$$

$$= \left(\int_{\alpha}^{\tau}dt\right)^{\frac{\ell}{\ell+km}}\left(\int_{\alpha}^{\tau}q(t)\,|\,x'(t)\,|^{\frac{\ell+km}{k}}\,dt\right)^{\frac{km}{\ell+km}}$$

$$\int_{\alpha}^{\tau}(q(t))^{\frac{km}{\ell+km}}\,|\,x'(t)\,|^{m}\,dt \le (\tau-\alpha)^{\frac{\ell}{\ell+km}}\left(\int_{\alpha}^{\tau}q(t)\,|\,x'(t)\,|^{\frac{\ell+km}{k}}\,dt\right)^{\frac{km}{\ell+km}} \tag{23}$$

Taking $\dfrac{\ell+km}{km}$ power of both sides in (23) we have

$$\left(\int_{\alpha}^{\tau}(q(t))^{\frac{km}{\ell+km}}\,|\,x'(t)\,|^{m}\,dt\right)^{\frac{\ell+km}{km}} \le (\tau-\alpha)^{\frac{\ell}{km}}\int_{\alpha}^{\tau}q(t)\,|\,x'(t)\,|^{\frac{\ell+km}{k}}\,dt \tag{24}$$

Since $y(t) = (q(t))^{\frac{km}{\ell+km}}\,|\,x'(t)\,|^{m}$. Then, integrating both sides with respect to t over $[\,\alpha,\tau\,]$ hence (24) becomes

$$\left(y(\tau)\right)^{\frac{\ell+km}{km}} \le (\tau-\alpha)^{\frac{\ell}{km}}\int_{\alpha}^{\tau}q(t)\,|\,x'(t)\,|^{\frac{\ell+km}{k}}\,dt \tag{25}$$

Combining (22) and (25) , we obtain

$$\int_{\alpha}^{\tau}q(t)\,|\,x(t)\,|^{\frac{\ell}{k}}\,|\,x'(t)\,|^{m}\,dt \le \frac{km}{km+\ell}(\tau-\alpha)^{\frac{\ell}{k}}\int_{\alpha}^{\tau}q(t)\,|\,x'(t)\,|^{\frac{km+\ell}{k}}\,dt. \tag{26}$$

Remark 2: For $\alpha=0$, $\tau=a$ and $k,m,q(t)=1$, Theorem 7 reduces to Theorem 3. Also, for $\alpha=0$, $m=n$ and $\ell=1$, Theorem 7 reduces to Theorem 6.

Conclusions

This work is Hua-type Opial inequality by means of extension. For further reference on time-scale, see Rauf and Anthinio (2017). Results therein are notable and useful in the concept of inequalities.

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